MYSURU ROYAL INSTITUTE OF TECHNOLOGY

DEPARTMENT OF PHYSICS

Applied Physics for EEE Stream : 2024-25

Applied Physics Notes

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Applied Physics for EEE Stream : 2022-23

Module – 1 : Quantum Mechanics

Syllabus:

Quantum Mechanics: **8 hrs**

deBroglie Hypothesis and Matter Waves, deBroglie wavelength and derivation of expression by analogy, Phase Velocity and Group Velocity, Heisenberg's Uncertainty Principle and its application (Non - existence of electron inside the nucleus - Non Relativistic), Principle of Complementarity.

Wave Function, Time independent Schrödinger wave equation (Derivation), Physical Significance of a wave function and Born Interpretation, Expectation value, Eigen functions and Eigen Values, Particle inside one dimensional infinite potential well, Quantization of Energy States, Waveforms and Probabilities. Numerical Problems.

Pre requisite: Wave–Particle dualism

Self-learning: deBroglie Hypothesis

Quantum Mechanics

The branch of mechanics that deals with the mathematical description of the motion and interaction of subatomic particles, incorporating the concepts of quantisation of energy, wave – particle duality, the uncertainty principle and the corresponding principles.

Introduction:

At the end of 19th century and in the beginning of 20th century, many new phenomena such as photoelectric effect, Compton Effect, pair production, Zeeman Effect, radiation effects, nuclear radiations etc., were discovered. Since classical mechanics fails to explain the above phenomena, a new physics known as modern physics was developed on the basis of quantum theory of radiation. In order to explain the distribution of energy in the blackbody radiation Planck introduced the concept of quantum theory of radiation in 1900.

Radiation:

Radiation is defined as the energy that travels through space or matter in the form of a particle or wave.

In physics, radiation is the emission or transmission of energy in the form of waves or particles through space or a material medium.

It is a process of transmission of energy from one place to another without the aid of any intervening medium.

Light and heat are types of radiation. Heat radiation is also called thermal radiation.

OR

"Radiation is the emission of electromagnetic waves by matter when supplied with appropriate amount of energy".

A photon is the smallest discrete packet or quantum of electromagnetic radiation. It is the basic unit of all light.

The basic properties of photons are:

- A packet or bundle of energy is called a photon
- The energy of a photon is $E = h\nu = hc/\lambda$
- The momentum of the photon is $p = E/c = h/\lambda$
- Photon can carry energy and momentum which are dependent on the frequency.
- The rest mass of the photon is zero and hence they can exist only moving states.
- They are charge less particles and are not affected by either electric or magnetic field,
- They are stable and having integral spin (spin-1 particles) which make them as bosons.
- They can have interactions with other particles such as electrons, protons, neutrons, etc.,
- They can travel with the speed of light in free space or vacuum.

Planck's Quantum Theory of Radiation:

In the year 1900, Max Planck propounded the quantum theory of radiation. According to this theory, the emission and absorption of radiation is not continuous but it is in the form of packets of definite and discrete set of energy. Each packet is called the quanta or photon which has a definite energy and definite momentum whose value is proportional to the frequency of radiation. If 'ν' is the frequency of radiation, the energy of each photon is given by,

$$E \propto \nu \qquad \text{or} \qquad E = h\nu = \frac{hc}{\lambda}$$

$$\text{But,} \quad E = mc^2 = \frac{hc}{\lambda} \qquad \text{or} \quad mc = \frac{h}{\lambda}$$

If a photon of mass ‘m’ moving with a velocity ‘c’, then its momentum p = mc

$$\therefore p = \frac{h}{\lambda} \quad \text{or} \quad \lambda = \frac{h}{p}$$

But, if photon of mass ‘m’ moving with a velocity ‘v’, then its momentum is p = mv

$$\therefore p = \frac{h}{\lambda} \quad \text{or} \quad \lambda = \frac{h}{p} = \frac{h}{mv}$$

Where ‘h’ is a universal constant, called the Planck's constant. Its value is 6.625 x 10^{-34} Js, c is the speed of light.

Matter Waves

Dual Nature of Matter:

The wave theory of electromagnetic radiation satisfactorily explains the phenomena of reflection, refraction, interference, diffraction and polarization. But it failed to explain the phenomena of Photoelectric Effect, Compton Effect.

On the other hand, they were explained on the basis of quantum theory of radiation. According to which a beam of light of frequency ν consists of small packets each having energy hν called photon or quanta.

Sometimes these photons behave like a waves and sometimes like a corpuscles i.e., particles. Thus radiation have dual nature i.e., wave and particle or quantum nature.

Matter waves and their characteristics properties

In 1924 Louis de Broglie suggested that the particles like protons, electrons, & neutrons in motion exhibit characteristic properties of waves. Thus a moving particle can be associated with a wave or a wave can guide the motion of the particle. Hence the waves associated with the moving particles are known as de-Broglie waves or matter waves.

According to de-Broglie hypothesis, a particle of mass ‘m’ moving with velocity ‘v’ is associated with the wave. This wave is called matter wave. The wavelength of matter wave in terms of its momentum ‘p’ is,

$$\lambda = \frac{h}{mv} = \frac{h}{p}$$

1. Matter waves are the waves associated with moving particles.
2. Lighter the particles, greater is the wavelength associated with it, because here

$$\lambda \propto 1/m$$

3. Greater the velocity of the particle, smaller is the wavelength associated with the particle.

$$\lambda \propto 1/\text{v}$$

4. Matter waves are not electromagnetic waves. Since they don’t depends on the charge of the particle.

5. The velocity of the matter waves is not constant. But it depends on the velocity of the particle.
6. Light wave has got same velocity, for all wavelengths. But in case of matter waves, the velocity is inversely proportional to the wavelength.
7. It is not possible to determine the exact position and momentum of a moving particle simultaneously.
8. Matter waves are also called as de-Broglie waves (or) pilot waves.

Note: (i) : If a particle of mass ‘m’ moving with a velocity ‘v’, then its kinetic energy

$$E = \frac{1}{2} mv^2 \quad \text{or} \quad m^2v^2 = 2mE \quad \text{or} \quad mv = \sqrt{2mE}$$

$\therefore \ \lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mE}}$ is the expression for deBroglie wavelength in terms of kinetic energy E.

(ii) : we know that, if an electron is accelerated under the potential difference of V, then the energy acquired by it will be ‘eV’, then

$$\therefore eV = \frac{1}{2}mv^2 = \frac{1}{2}\frac{m^2v^2}{m} = \frac{p^2}{2m}$$

$$or \quad p^2 = 2meV \quad \text{or} \quad p = \sqrt{2meV}$$

$$\therefore \ \lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV}}$$

This is the expression for deBroglie wavelength in terms of accelerating potential V.

Problems:-

(1) Calculate the de-Broglie wavelength associated with an electron having a kinetic energy of 100 eV.

Data:- $\lambda = ?$, E=100eV=100 x 1.602 x10^{-19}J

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{6.63\times10^{-34}}{\sqrt{2\times9.11\times10^{-31}\times100\times1.602\times10^{-19}}} = 1.228\times10^{-10}m = 1.228\,\text{A}^{\circ}$$

(2) (May22) Evaluate the de-Broglie wavelength of Helium Nucleus accelerated through a potential difference of 500 V.

Data:- $\lambda = ?$, V=500V, m=4m_p=4x1.673x10^{-27}kg

$$\lambda = \frac{h}{\sqrt{2meV}} = \frac{6.63\times10^{-34}}{\sqrt{2\times4\times1.673\times10^{-27}\times1.602\times10^{-19}\times500}} = 6.403\times10^{-13}m = 6403\text{A}^{\circ}$$

(3) Calculate the de-Broglie wavelength of an electron accelerated under a potential difference of 100 V.

Soln.: $\lambda = ?$, V=100V

$$\lambda = \frac{h}{\sqrt{2meV}} = \frac{6.63\times10^{-34}}{\sqrt{2\times9.11\times10^{-31}\times1.602\times10^{-19}\times100}} = 1.228\times10^{-10}m = 1.228\text{A}^{\circ}$$

(4) Compute the de-Broglie wavelength for a neutron moving with one tenth part of the velocity of light. Given mass of neutron=1.674x10^{-27}kg.

Soln.: $\lambda = ?,\ v = \frac{c}{10} \quad and\ m_n = 1.674\times10^{-27}\,kg$

$$\lambda = \frac{h}{mv} = \frac{6.63\times10^{-34}}{1.674\times10^{-27}\times\frac{3\times10^{8}}{10}} = 1.320\times10^{-14}\,m$$

(5) Find the KE of a neutron which has a wavelength of 3Å. Given its mass=1.674x10^{-27}kg.

$$\lambda = \frac{h}{\sqrt{2mE}} \quad \Rightarrow \quad E = \frac{h^2}{2m\lambda^2}$$

$$= \frac{\left(6.63\times10^{-34}\right)^2}{2\times1.674\times10^{-27}\times\left(3\times10^{-10}\right)^2} = 1.46\times10^{-21}\,J = 9.114\times10^{-3}\,eV$$

(6) Estimate the potential difference through which a proton is needed to be accelerated so that its de- Broglie wavelength becomes equal to 1 Å, given that its mass is1.673x10^{-27}kg.

$$\lambda = \frac{h}{\sqrt{2meV}} \quad \Rightarrow \quad V = \frac{h^2}{2me\lambda^2}$$

$$= \frac{\left(6.63\times10^{-34}\right)^2}{2\times1.673\times10^{-27}\times1.602\times10^{-19}\times\left(1\times10^{-10}\right)^2} = 0.082V.$$

(7) The kinetic energy of an electron is equal to the energy of photon with a wavelength of 560nm. Calculate the de-Broglie wavelength of the electron.

$$E = h\nu = \frac{hc}{\lambda},\ \lambda_p = 560nm = 560\times10^{-9}\,m,\ \lambda = ?$$

$$E = \frac{hc}{\lambda_p} = \frac{6.63\times10^{-34}\times3\times10^{8}}{560\times10^{-9}} = 3.552\times10^{-19}\,J = 2.217eV$$

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{6.63\times10^{-34}}{\sqrt{2\times9.11\times10^{-31}\times3.552\times10^{-19}}} = 8.25\times10^{-10}\,m$$

(8) Calculate the de-Broglie wavelength associated with an electron having a kinetic energy of 100 eV.

Soln.: $\lambda = ?$, E=100eV=100 x 1.602 x10^{-19}J

$$\lambda = \frac{h}{\sqrt{2mE}}$$

$$\lambda = \frac{6.63\times10^{-34}}{\sqrt{2\times9.11\times10^{-31}\times100\times1.602\times10^{-19}}}$$

$$\lambda = 1.228\times10^{-10}\,m$$

(9) (May22) Evaluate the de-Broglie wavelength of Helium Nucleus accelerated through a potential difference of 500 V.

Soln.: $\lambda = ?$, V = 500V, m = 4m$_p$ = 4x1.673x10^{-27}kg

$$\lambda = \frac{h}{\sqrt{2meV}}$$

$$= \frac{6.63\times10^{-34}}{\sqrt{2\times4\times1.673\times10^{-27}\times1.602\times10^{-19}\times500}}$$

$$= 6.403\times10^{-13}m$$

(10) Calculate the de-Broglie wavelength of an electron accelerated under a potential difference of 100 V.

Soln.: $\lambda = ?$, V=100V

$$\lambda = \frac{h}{\sqrt{2meV}}$$

$$= \frac{6.63\times10^{-34}}{\sqrt{2\times9.11\times10^{-31}\times1.602\times10^{-19}\times100}}$$

$$= 1.228\times10^{-10}m$$

(11) Compute the de-Broglie wavelength for a neutron moving with one tenth part of the velocity of light. Given mass of neutron=1.674X 10^{-27}kg.

Soln.: $\lambda = ?,\ v = \frac{c}{10}\quad and\ m_n = 1.674\times10^{-27}kg$

$$\lambda = \frac{h}{mv}$$

$$= \frac{6.63\times10^{-34}}{1.674\times10^{-27}\times\frac{3\times10^8}{10}}$$

$$= 1.320\times10^{-14}m$$

(12) Find the KE of a neutron which has a wavelength of 3Å. Given its mass=1.674X 10^{-27}kg.

$$\lambda = \frac{h}{\sqrt{2mE}} \Rightarrow$$

$$E = \frac{h^2}{2m\lambda^2}$$

$$= \frac{(6.63\times10^{-34})^2}{2\times1.674\times10^{-27}\times(3\times10^{-10})^2}$$

$$= 1.46\times10^{-21}J$$

(13) Estimate the potential difference through which a proton is needed to be accelerated so that its de- Broglie wavelength becomes equal to 1Å, given that its mass is1.673X 10^{-27} kg.

$$\lambda = \frac{h}{\sqrt{2meV}} \Rightarrow$$

$$V = \frac{h^2}{2me\lambda^2}$$

$$= \frac{\left(6.63\times10^{-34}\right)^2}{2\times1.673\times10^{-27}\times1.602\times10^{-19}\times\left(1\times10^{-10}\right)^2}$$

$$= 0.082V.$$

(14) The kinetic energy of an electron is equal to the energy of photon with a wavelength of 560nm. Calculate the de-Broglie wavelength of the electron.

$$E = h\nu = \frac{hc}{\lambda}, \lambda_p = 560nm = 560\times10^{-9}m, \lambda \; = ?$$

$$E = \frac{hc}{\lambda_p}$$

$$E = \frac{6.63\times10^{-34}\times3\times10^{8}}{560\times10^{-9}}$$

$$E = 3.552\times10^{-19}J$$

$$\lambda = \frac{h}{\sqrt{2mE}}$$

$$\lambda = \frac{6.63\times10^{-34}}{\sqrt{2\times9.11\times10^{-31}\times3.552\times10^{-19}}}$$

$$\lambda = 8.25\times10^{-10}m$$

Wave Packet: A wave packet consisting of waves of slightly differing wavelengths may represent the moving particle. Superposition of these waves constituting the wave packet results in the net amplitude being modified, thereby defining the shape of the wave group.

A wave is represented by the formula

$$y = ASin(\omega t - kx)$$

Where y is the displacement at any instant t, A is the amplitude of vibration, ω is the angular frequency (ω = 2πν) and k is the wave vector (k = 2π/λ).

Phase Velocity (v_{Phase} or v_p)

A point marked on a wave can be regarded as representing a particular phase for the wave at that point. The velocity with which such a point would propagate is known as phase velocity (or) wave velocity. It is represented by

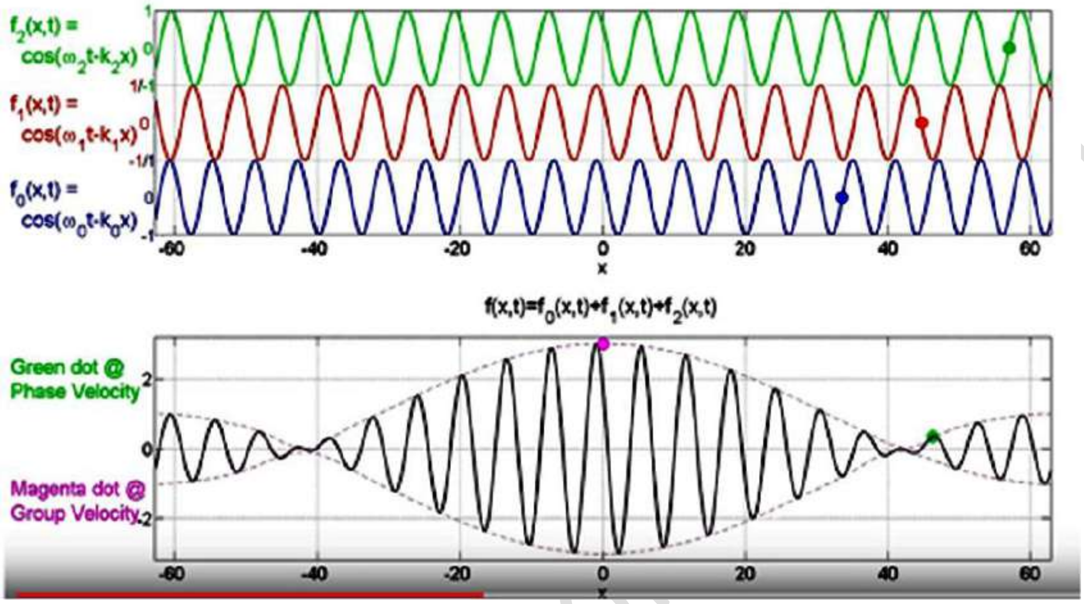
$$v_{phase} \quad \text{or} \quad v_p = \frac{\omega}{k}$$

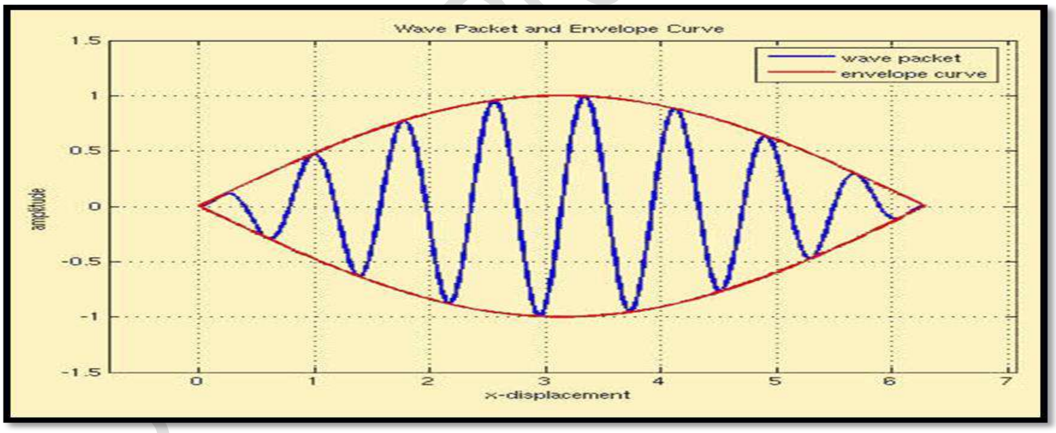
where, ω is angular frequency and k is the propagation constant or wave number

Group Velocity (v_{group} or v_g)

The velocity with which the resultant envelops of the group of waves travels is called group velocity.
It is denoted by v_g or v_{group} and is equal to the particle velocity v.

$$v_{group} \quad or \quad v_g = \frac{d\omega}{dk}$$





Relation between Group Velocity v_g and Phase Velocity v_p

The equations for group velocity and phase velocity are given by,

$$v_g = \frac{dw}{dk} \qquad \rightarrow (1)$$

$$v_p = \frac{\omega}{k} \qquad \rightarrow (2)$$

where ω is the angular frequency of the wave and k is the propagation constant or wave vector.

$$\therefore \omega = v_p k \qquad \rightarrow (3)$$

$$\frac{dw}{dk} = \frac{d\left(\mathrm{v}_p k\right)}{dk} = \mathrm{v}_p + k\frac{dv_p}{dk} \qquad \rightarrow (4)$$

$$\text{But,} \quad k\frac{dv_p}{dk} = k\frac{dv_p}{d\lambda}\,\mathrm{x}\,\frac{d\lambda}{dk} \qquad \rightarrow (5)$$

$$\text{we know that,} \quad k = \frac{2\pi}{\lambda} \quad or \quad \lambda = \frac{2\pi}{k}$$

$$\text{and} \quad \frac{dk}{d\lambda} = 2\pi\left(\frac{-1}{\lambda^2}\right) \qquad or \qquad \frac{d\lambda}{dk} = -\frac{\lambda^2}{2\pi}$$

$$\therefore \text{eqn (5) becomes,} \quad k\frac{dv_p}{dk} = \frac{2\pi}{\lambda}\,\mathrm{x}\,\frac{-\lambda^2}{2\pi}\,\mathrm{x}\,\frac{dv_p}{d\lambda}$$

$$or \qquad k\frac{dv_p}{dk} = -\lambda\,\mathrm{x}\,\frac{dv_p}{d\lambda}$$

On substituting this values in equation (4) we get,

$$\mathrm{v}_g = \mathrm{v}_p - \lambda\frac{dV_p}{d\lambda}$$

$$or \qquad \mathrm{v}_{group} = \mathrm{v}_{phase} - \lambda\left(\frac{d\mathrm{v}_{phase}}{d\lambda}\right)$$

This is the relation between group velocity and phase velocity.

Heisenberg’s uncertainty principle

According to this principle “It is impossible to determine precisely and simultaneously the values of both the members of the pair of physical variables, which describe the motion of the atomic system”. Such variables are called canonically conjugate variables.

Example: Position and momentum, energy and time etc.,

Statement: *“it is impossible to determine simultaneously both position and momentum of a moving particle accurately at same time. The product of uncertainty in these quantities is always greater than or equal to h/4π”.*

If Δx and ΔP_x are the uncertainties in the measurement of position and momentum of a particle, then

$$\Delta x.\,\Delta p_x \geq \frac{h}{4\pi}$$

If Δx is small, ΔP_x will be large and vice versa. That is if one quantity is measured accurately, the other quantity becomes less accurate.

Similarly the other uncertainty relations for other physical variables pair are,

$$\Delta E.\Delta t \geq \frac{h}{4\pi}$$

$$\Delta L.\Delta \theta \geq \frac{h}{4\pi}$$

Applications of Uncertainty Principle:

Non-existence of electrons in the nucleus and its implications non-relativistic approach

According to theory of relativity, if a particle of mass moving with a velocity v ,then the energy E and momentum p of the particle are expressed as,

$$E = mc^2 \quad \text{and} \quad p = mv \qquad \rightarrow (1)$$

where m is the relativistic mass of a particle moving the a velocity v and the expression for it in terms of rest mass m_0 can be written as,

$$m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \rightarrow (2)$$

$$m^2 = \frac{m_0^2}{1-\frac{v^2}{c^2}} = \frac{m_0^2 c^2}{c^2 - v^2}$$

$$m^2(c^2 - v^2) = m_0^2 c^2 \quad \text{or} \quad m^2c^2 - m^2v^2 = m_0^2c^2 \quad \text{x } c^2$$

$$m^2c^4 - m^2v^2c^2 = m_0^2c^4$$

Since E = mc^2 and p = mv, the above equation becomes

$$E^2 - p^2c^2 = m_0^2c^4 \quad \text{or} \quad E^2 = p^2c^2 + m_0^2c^4$$

$$E = c\sqrt{p^2 + m_0^2c^2} \quad \text{or} \quad E = c\left(p^2 + m_0^2c^2\right)^{1/2} \qquad \rightarrow (3)$$

According to Heisenberg's uncertainty principle we have,

$$\Delta x.\Delta p_x \geq \frac{h}{4\pi} \quad \text{or} \quad \Delta p_x \geq \frac{h}{4\pi.\Delta x} \qquad \rightarrow (4)$$

We know that the size / diameter of the nucleus is of the order of 10^{-14} m. If an electron is to exist inside the nucleus, then the uncertainty in its position Δx must not exceed the size of the nucleus,

$$i.e., \quad \Delta x \leq 10^{-14} m$$

Using Δx in equation (5) we have,

$$\Delta p_x \geq \frac{h}{4\pi.\Delta x} \geq \frac{6.63\times10^{-34}}{4\pi\times10^{-14}} \geq 0.5\times10^{-20} \quad \text{Ns}$$

$$\therefore \quad p_x \geq 0.5\times10^{-20} \quad Ns$$

This is the uncertainty in momentum of an electron and it is equal to the momentum of the electron inside the nucleus,

Using momentum value P_x in an equation (3) we get,

$$E = 3x10^8\left((0.5x10^{-20})^2 + (9.11x10^{-31})^2 x(3x10^8)^2\right)^{1/2}$$

$$E = 1.5x10^{-12}\,J = \frac{1.5x10^{-12}}{1.602x10^{-19}}\,eV$$

$$E = 9.4x10^{6}\,eV \approx 9.4MeV$$

An electron may exist inside the nucleus if its energy is equal to or greater than 9.4 MeV. But, the experimental investigations on β-decay say that kinetic energy of the β–particles is 3 to 4 MeV. This clearly indicates that, electrons cannot exist within the nucleus.

$$n \rightarrow p + e^{-1} + \bar{v}\ (energy) \qquad \text{or} \qquad p \rightarrow n + e^{+1} + v\ (energy)$$

Principle of Complementarity

Statement: Bohr stated as "In a situation where the wave aspect of a system is revealed, its particle aspect is concealed; and in a situation where the particle aspect is revealed, its wave aspect is concealed. Revealing both simultaneously is impossible; the wave and particle aspects are complementary."

Explanation: We know that the consequence of the uncertainty principle is both the wave and particle nature of the matter cannot be measured simultaneously. In other words, we cannot precisely describe the dual nature of Light.

- If an experiment is designed to measure the particle nature of the matter, during this experiment, errors of measurement of both position and the time coordinates must be zero and hence the momentum, energy and the wave nature of the matter are completely unknown.

- Similarly, if an experiment is designed for measuring the wave nature of the particle, then the errors in the measurement of the energy and the momentum will be zero, whereas the position and the time coordinates of the matter will be completely unknown. unknown.

From the above explanation, we can conclude that, when the particle nature of the matter is measured or displayed, the wave nature of the matter is necessarily suppressed and vice versa.

Problems:

1. If the group velocity of a particle is $3x10^6$ m/s, calculate its phase velocity. (Given, c = $3x10^8$ m/s).

$$Given,\quad v_g = 3\times10^6\,m/s,\quad c = 3\times10^8\ m/s,\quad v_p = ?$$

$$w.k.t; \qquad V_{group}V_{phase} = c^2$$

$$or\quad V_{phase} = \frac{c^2}{V_{group}} = \frac{(3\times10^8)^2}{(3\times10^6)} = \frac{9\times10^{16}}{3\times10^6} = 3\times10^{10}\,m/s$$

2. Calculate the de-Broglie wavelength associated with a proton moving with a velocity equal to 1/20 th of the velocity of light.

$$Given,\quad v = \frac{1}{20}\times c = \frac{1}{20}\times3\times10^8,\quad m_p = 1.67\times10^{-27}\,kg,\quad \lambda_p = ?$$

$$w.k.t,\quad \lambda_p = \frac{h}{mv}$$

$$\lambda_p = \frac{6.63\times10^{-34}}{1.67\times10^{-27}\times\frac{1}{20}\times3\times10^8} = \frac{20\times6.63\times10^{-34}}{1.67\times10^{-27}\times3\times10^8} = 2.646\times10^{-14}\,m$$

3. Find the kinetic energy and group velocity of an electron with de-Broglie wavelength of 0.2 nm.

$$Given,\ \ \lambda = 0.2\,nm = 0.2\times10^{-9}m,\ \ K.E = ?,\ \ V_{group} = ?,\ \ m_e = 9.1\times10^{-31}kg$$

$$w.k.t;\quad \lambda = \frac{h}{p}$$

$$\therefore\ p = \frac{h}{\lambda} = \frac{6.63\times10^{-34}}{0.2\times10^{-9}} = 3.315\times10^{-24}kg - m/s$$

$$K.E = E = \frac{p^2}{2m} = \frac{(3.315\times10^{-24})^2}{2\times9.1\times10^{-31}} = 6.038\times10^{-18}J$$

$$V_g\ or\ V_{group} = \frac{p}{m} = \frac{3.315\times10^{-24}}{9.1\times10^{-31}} = 3642857.143 = 3.64\times10^{6}m/s$$

5. If the uncertainty in the position of an electron is 4x10^{-10} m, calculate the uncertainty in its momentum.

$$Given,\ \ \Delta x = 4\times10^{-10}m,\quad \Delta P_x = ?$$

$$\text{w}.k.t\quad \Delta x.\Delta p_x \geq \frac{h}{4\pi}$$

$$\therefore\ \Delta p_x \geq \frac{h}{4\pi\,\Delta x} \geq \frac{6.63\times10^{-34}}{4\times3.14\times4\times10^{-10}} \geq 1.318\times10^{-25}kgs^{-1}$$

6. In a simultaneous measurement of position and velocity of an electron moving with a speed of 6x10^{5} m/s. Calculate the highest accuracy with which its position could be determined if the inherent error in the measurement of the velocity is 0.01% for the speed stated.

$$Given,\ \ v = 6\times10^{5}m/s,\ \ \%\ of\ \ error\ \ in\ velocity = 0.01\%,\quad \Delta x = ?$$

$$percentage\ \ of\ uncerta\,\mathrm{int}\ y\ \ in\ \ velocity = \frac{uncerta\,\mathrm{int}\ y\ in\ velocity}{velocity}\times100$$

$$uncertainity\ \ in\ \ velocity = \frac{percentage\ of\ uncertainity\ \ in\ \ velocity}{100}\times velocity$$

$$\Delta v = \frac{\%\ of\ error\ in\ the\ measurment\ of\ velocity}{100}\times v$$

$$\Delta v = \frac{0.01}{100}\times6\times10^{5} = 60ms^{-1}$$

$$\text{w}.k.t;\quad \Delta x.\Delta p_x \geq \frac{h}{4\pi}$$

$$\Delta x \geq \frac{h}{4\pi.m.\Delta v_x} \geq \frac{6.63\times10^{-34}}{4\times3.142\times9.1\times10^{-31}\times60} \geq 9.66\times10^{-7}m = 966\ nm$$

7. An electron has a speed of 300m/s accurate to 0.01% with what fundamental accuracy can we locate the position of the electron.

Given, $V = 300$ m/s, $\Delta x = ?$, $\Delta v = ?$ and % of accuracy in speed = 0.01

$$\Delta v = 0.01\%\ of\ v \quad \text{or} \quad \Delta v = \frac{0.01}{100}\times v = \frac{0.01}{100}\times 300 = 0.03\, m/s$$

$$\Delta v = 3\times 10^{-2}\, m/s$$

w.k.t

$$\Delta x.\Delta p_x \geq \frac{h}{4\pi} \quad \text{or} \quad \Delta x \geq \frac{h}{4\pi\,\Delta p_x} = \frac{h}{4\pi.m.\Delta v}$$

$$\therefore \Delta x \geq \frac{6.63\times 10^{-34}}{4\times 3.142\times 9.1\times 10^{-31}\times 3\times 10^{-2}} = 1.932\times 10^{-3}\, m$$

$\therefore$ The maximum accuracy with which the electron can be located is 1.932 x 10^{-3} m.

8. The speed of electron is measured to within an uncertainty of $1\text{x}10^4$ m/s. What is the minimum space required by the electron to be confined in an atom?

Given, $\Delta v = 1\times 10^4\, m/s, \Delta x = ?$

w.k.t; $\Delta x.\Delta p_x \geq \frac{h}{4\pi}$

$$\therefore\ \Delta x \geq \frac{h}{4\pi\,\Delta p_x} = \frac{h}{4\pi.m.\Delta v} \geq \frac{6.63\times 10^{-34}}{4\times 3.142\times 9.1\times 10^{-31}\times 1\times 10^4} = 5.797\times 10^{-9}\, m$$

$$\Delta x \geq 57.97\times 10^{-10}\, m = 57.97\ \overset{0}{A}$$

9. The position and momentum of 1keV electron are simultaneously determined and if its position is located within 1Å. What is the minimum percentage of uncertainty in its momentum?

Given, $\text{E} = 1\text{keV},\ \text{E} = 1\text{x}10^3\text{eV},\ \text{E} = 1\text{x}10^3\text{x}1.602\text{x}10^{-19}\ \text{J},\ \text{E} = 1.602\text{x}10^{-16}\ \text{J}$

$$\Delta x = 1\overset{0}{A} = 1\times 10^{-10}\, m \qquad \Delta P = ?\ ,\ \frac{\Delta P}{P}\times 100 = ?$$

$$w.k.t; \quad \Delta x.\Delta p_x \geq \frac{h}{4\pi}$$

$$\therefore \quad \Delta P_x \geq \frac{h}{4\pi\,\Delta x} \geq \frac{6.63\times 10^{-34}}{4\times 3.142\times 10^{-10}} = 5.275\times 10^{-25}\ kgms^{-1}$$

we have the equation for momentum,

$$P = \sqrt{2mE} = \sqrt{2\times 9.1\times 10^{-31}\times 1.602\times 10^{-16}}$$

$$P = 1.707\times 10^{-23}\, kg\, ms^{-1}$$

$$Percentage\ of\ uncertainity\ in\ momentum = \frac{Uncertainity\ in\ momentum}{momentum}\times 100$$

$$Percentage\ of\ uncertainity\ in\ momentum = \frac{5.275\times 10^{-25}}{1.707\times 10^{-23}}\times 100 = 3.087 = 3.1$$

10. The inherent uncertainty in the measurement of time spent by Iridium -191 nuclei in the excited state is found to be $1.4\text{x}10^{-10}$s. Estimate the uncertainty that results in its energy in the excited state.

$$Given,\ \Delta t = 1.4\times10^{-10}s,\quad \Delta E = ?$$

$$\text{w.k.t;}\quad \Delta E.\Delta t \geq \frac{h}{4\pi}$$

$$\therefore\quad \Delta E \geq \frac{h}{4\pi.\Delta t} \geq \frac{6.63\times10^{-34}}{4\times3.142\times1.4\times10^{-10}} = 3.768\times10^{-25}J$$

$$\Delta E = \frac{3.768\times10^{-25}}{1.602\times10^{-19}}eV = 2.353\times10^{-6}eV$$

11. The average time that an atom retains excess excitation energy before re- emitting it in the form of electromagnetic radiation is 10^{-8}s. Calculate the limit of accuracy with which the excitation energy of the emitted radiation can be determined.

(OR)

What is the minimum uncertainty in the energy state of an atom if an electron remains in this state for 10^{-8} seconds?

$$Given,\ \Delta t = 1.4\times10^{-10}s,\quad \Delta E = ?$$

$$\text{w.k.t;}\quad \Delta E.\Delta t \geq \frac{h}{4\pi}$$

$$\therefore\quad \Delta E \geq \frac{h}{4\pi.\Delta t} \geq \frac{6.63\times10^{-34}}{4\times3.142\times10^{-8}} = 5.275\times10^{-27}J$$

$$\Delta E = \frac{5.275\times10^{-27}}{1.602\times10^{-19}}eV = 3.292\times10^{-8}eV$$

12. An electron is confined to a box of length 10^{-8}m. Calculate the minimum uncertainty in its velocity.

$$Given,\ \Delta v = ?\quad \Delta x = 10^{-8}m$$

$$w.k.t;\quad \Delta x.\Delta p_x \geq \frac{h}{4\pi}$$

$$or\quad \Delta P_x \geq \frac{h}{4\pi\,\Delta x}\quad \text{or}\quad m.\Delta v_x = \frac{h}{4\pi.\Delta x}$$

$$\therefore\ \Delta v_x = \frac{h}{4\pi.m.\Delta x} \geq \frac{6.63\times10^{-34}}{4\times3.142\times10^{-8}\times9.1\times10^{-31}} = 5797.03\ m/s$$

$$\Delta v = 5800\ m/s$$

(13) If the kinetic energy of an electron known to be about 1eV, must be measured to within 0.0001eV, What accuracy can its position be measured simultaneously?

Soln.: E = 1eV = $1.602\text{x}10^{-19}$J, ΔE = 0.0001eV = $0.0001\text{x}1.602\text{x}10^{-19}$J, Δx = ?

$$\Delta P\Delta X = \frac{h}{4\pi}\quad and\quad P = \sqrt{2mE} \Rightarrow$$

$$\Delta P = \frac{1}{2}(2mE)^{-1/2}2m\Delta E$$

$$= \sqrt{\frac{m}{2E}}\Delta E$$

$$\therefore \Delta X = \frac{h}{4\pi}\frac{1}{\Delta E}\sqrt{\frac{2E}{m}}$$

$$\therefore \Delta X = \frac{6.63\times10^{-34}\times\sqrt{2\times1.602\times10^{-19}}}{4\times\pi\times0.0001\times1.602\times10^{-19}\times\sqrt{9.1\times10^{-31}}}$$

$$=1.95\times10^{-6}m$$

(14) Uncertainty in time of an exited atom is about 10^{-8}s. What are the uncertainties in energy and in frequency of the radiation?

Soln.: $\Delta t = 10^{-8}$s, ΔE = ?, $\Delta\nu$ = ?

$$\Delta E\Delta t = \frac{h}{4\pi}$$

$$\Delta E = \frac{h}{4\pi.\Delta t}$$

$$= \frac{6.63\times10^{-34}}{4\times\pi\times10^{-8}}$$

$$= 5.23\times10^{-27}J$$

$$E = h\nu$$

$$\therefore \Delta E = h\Delta\nu$$

$$\therefore \Delta\nu = \frac{\Delta E}{h}$$

$$= \frac{5.23\times10^{-27}}{6.63\times10^{-34}}$$

$$= 7.89\times10^{6}Hz$$

(3) (May22) The position and momentum of an electron with energy 0.5 keV is found with a minimum percentage uncertainty in momentum. Find its uncertainty if the measurement of position has a uncertainty of 0.5Å.

Soln.: E=0.5keV=$0.5\times10^{3}\times1.602\times10^{-19}$J, Percentage uncertainty in momentum = ?,

$\Delta x = 0.5\times10^{-10}$ m.

$$\Delta P\Delta X = \frac{h}{4\pi} \quad and \quad P=\sqrt{2mE}$$

$$\Delta P = \frac{h}{4\pi\times\Delta X} \quad and \quad P=\sqrt{2mE}$$

$$\Delta P = \frac{6.63\times10^{-34}}{4\pi\times0.5\times10^{-10}} \quad and \quad P=\sqrt{2\times9.1\times10^{-31}\times0.5\times10^{3}\times1.602\times10^{-19}}$$

$$\Delta P = 1.055\times10^{-24} \quad and \quad P=1.207\times10^{-23}$$

Questions

Short Answer Questions

1. What is Planck's law of radiation?
2. List out the characteristics of matter waves.
3. State and explain Planck's law of radiation.
4. What are matter waves?
5. Define phase velocity and group velocity.
6. Obtain an expression for deBroglie wavelength.
7. State and Explain deBroglie hypothesis.
8. What are matter waves and mention their properties?
9. Derive the relation between Phase velocity and Group velocity.
10. What is de-Broglie concept of matter wave? Explain the characteristics of matter wave.
11. Define phase velocity and group velocity & obtain a relation between them.
12. State Heisenberg Uncertainty Principle.
13. Show that the electron does not exists inside the nucleus of an atom.
14. Explain the principle of complementarity.
15. State and Explain Heisenberg's uncertainty principle.
16. Explain why electron cannot exist inside the nucleus?
17. State and Explain the Principle of Complementarity.

Problems:

1. Compare the energy of photon with that of an electron when both are associated with a wavelength of 0.2 nm.

2. Calculate the deBroglie wavelength of a 1000 kg automobile travelling at 100 m/s and a 0.1 kg bullet travelling at 500 m/s.

3. A fast moving neutron is found to a have an associated deBroglie wavelength of $2x10^{-12}$ m. Find its kinetic energy and group velocity of the deBroglie waves using the relativistic change in mass. (Mass of neutron = $1.675x10^{-27}$ kg)

6. Calculate the deBroglie wavelength associated with an electron with a kinetic energy of 2000 eV.

7. Calculate the momentum of the particle and the deBroglie wavelength associated with an electron with a kinetic energy of 1.5 keV.

8. Calculate the wavelength associated with an electron having kinetic energy 100 eV.

9. Calculate deBroglie wavelength associated with electron carrying energy 2000 eV.

10. Find the energy of the neutron in eV whose deBrogle wavelength is 1 Å.

11. Calculate deBroglie wavelength associated with neutron of mass 1.674×10^{-27} kg with 1/10th part of speed of light.

12. Calculate deBroglie wavelength associated with electrons whose speed is 0.01 part of the speed of light.

13. What is the deBroglie wavelength of a proton whose energy is 3eV given mass of proton is 1.67×10^{-27} kg.

14. Find the kinetic energy and group velocity of an electron with deBroglie wavelength of 0.2 nm.

15. Calculate the deBroglie wavelength of particle of mass 0.65 MeV/c^2 has a kinetic energy 80 eV.

16. Find deBroglie wavelength of a particle of mass 0.58 MeV/c^2 has a kinetic energy 90 eV, where c is speed of light.

17. A particle of mass 940 MeV/c^2 has kinetic energy 0.5 keV. Find it deBroglie wavelength, where c is speed of light.

18. Find the deBroglie wavelength of an electron accelerated through a potential difference of 182 V and object of mass 1 kg moving with a speed of 1 m/s. Compare the results and comment.

19. The position and momentum of an electron with energy 0.5 keV are determined. What is the minimum percentage uncertainty in its momentum if the uncertainty in the measurement of position is 0.5Å?

20. The speed of electron is measured to within an uncertainty of 2.2×10^4 m/s in one dimension. What is the minimum width required by the electron to be confined in an atom?

Wave Function

Introduction:

In general, a wave is characterised by periodic variation in some physical quantity.

For example – pressure varies periodically in sound waves whereas electric and magnetic fields vary periodically in an electromagnetic wave. Similarly, whose periodic variations make up the matter wave is called wave function.

Wave Function:

The variable quantity that characterises the deBroglie wave is called wave function. Wave function in quantum mechanics accounts for the wave like properties of particle and is obtained by solving a fundamental equation called Schrödinger's equation.

The wave functions vary with respect to both position co-ordinates of the physical system and the time (x, y, z & t) is called total wave function.

It is denoted by the capital form of Greek letter 'Ψ'. If the wave function has variation only with position (x, y, z) it is denoted by the lower case Greek letter 'ψ'.

The total wave function can be mathematically represented by the equation

$$\Psi = Ae^{i(kx-\omega t)}$$

Where A is a constant and *w* is the angular frequency of the wave

The above equation can be written as

$$\Psi = Ae^{ikx}e^{-i\omega t}$$

Where, $\Psi = Ae^{ikx}$ is the space dependent wave function and is therefore time independent wave function. $e^{-i\omega t}$ is the time dependent wave function.

The wave function ψ is a measure of finding the particle at a particular position (x, y, z) and at time t.

The following are the basic properties of wave function:

1. The wave function ψ itself does not have any physical significance.
2. ψ is a positive or negative or complex quantity and hence it cannot be measured.
3. ψ is a function of space and time (r, t) coordinates and describes the behaviour of a single particle or photon and wave nature.
4. ψ is a large magnitude where the particle (Photon, electron etc.,) to be located and small at other places.
5. The probability of finding a particle at some point in space at time 't' is a positive value between 0 & 1; i.e., $|\psi|^2$ is real and +ve between 0 & 1.

Probability Density:

In classical mechanics, the square of wave amplitude associated with electromagnetic radiation is interpreted as measure of intensity. This suggests there will be a similar interpretation for de-Broglie waves associated with electron or any particle.

Let τ be a volume inside which a particle is present, but where exactly the particle is situated inside τ is not known

"If ψ is the wave function associated with the particle then the probability of finding the particle in certain volume $d\tau$ of τ is equal to $|\psi|^2\, d\tau$. So $|\psi|^2$ is called the probability density".

$$|\psi|^2\, d\tau$$

This interpretation was first given by Max Born in 1926.

If the value of $|\psi|^2$ is large at a point in a given time, then the probability of finding the particle at that point and time is more. If $|\psi|^2 = 0$, then the probability of finding the particle is zero or less.

Therefore the total wave function can be represented by the equation,

$$\Psi = Ae^{i(kx-\omega t)} \rightarrow (1)$$

where A is a constant, ω is angular frequency of the wave
The complex conjugate of Ψ is given by,

$$\Psi^* = Ae^{-i(kx-\omega t)} \rightarrow (2)$$

From equation (1) and (2), $\Psi\Psi^*$ is real and positive quantity which is called the probability density.

$$\text{i.e,} \quad |\psi|^2 = \psi\psi^* = A^2$$

Therefore $|\Psi|^2\, dx$ is the probability density in 1- dimension,

and $|\Psi|^2\, dv$ is the probability density in 3-dimension.

Normalization:

According to Born's interpretation the probability of finding the particle within an element of volume is $|\Psi|^2\, dv$, since the particle is certainly present somewhere inside the volume dv.

Therefore "The integral of the square of the wave function over the entire volume in space must be equal to unity" and mathematically it is represented as,

$$\int_{-\infty}^{\infty} |\psi|^2\, dv = 1$$

Where, the wave function satisfying the above relation is the normalized wave function.

Very often Ψ is not a normalized wave function. If this function Ψ is multiplied by a constant A, then the new wave function $A\Psi$ is also a solution of the wave equation. Hence the new wave function is a normalized wave function, if

$$\int_{-\infty}^{\infty} |A\psi^*||A\psi|\, dv = 1 \quad \text{or} \quad A^2\int_{-\infty}^{\infty} \psi\psi^*\, dv = 1$$

$$|A|^2 = \frac{1}{\int_{-\infty}^{\infty} \psi^* \psi \, dv}$$

Where $|A|^2$ is known as normalizing constant, the quantity $|A\Psi|^2$ represents probability. Therefore, the process of constructing AΨ from Ψ is called normalization of the wave function.

Limitations of wave function:

1. The wave function Ψ must be finite for all values of x, y, z and it is finite for a particular point.
2. Ψ must be single valued everywhere for each set of x, y, z and must have unique value.
3. Ψ must be continuous in all regions except where the potential energy V is infinite.
4. Ψ and its first derivatives dΨ/dx, dΨ/dy, dΨ/dz must be continuous and single valued everywhere.
5. Ψ must be normalised and in order that $|\Psi|^2$ dv over all space be a finite constant.

Expectation Value

In quantum mechanics, the expectation value is the probabilistic expected value of the result (measurement) of an experiment. It can be thought of as an average of all the possible outcomes of a measurement as weighted by their likelihood. Expectation value as such it is not the most probable value of a measurement. In the real sense the expectation value may have zero probability of occurring.

Let us consider a particle moving along the x axis. The result of a measurement of the position x is a continuous random variable. Consider a wave function $\Psi(x, t)$. The $|\Psi(x, t)|^2$ value is a probability density for the position observable and $|\Psi(x, t)|^2 dx$ is the probability of finding the particle between x and $x+dx$ at time t. Thus, if a measurement of position is repeated many times in an identical way on an identical particle in identical circumstances, many possible outcomes are possible and the expectation value of these outcomes is, according to the following equation

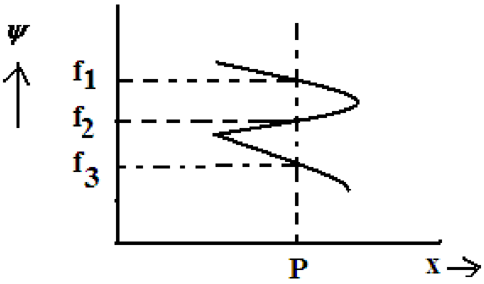
$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi(x, t)|^2 dx$$

Properties of Wave Functions:

A system is characterised by its position, energy, momentum etc,. In quantum mechanics, the state of a system is completely characterised by a wave function.

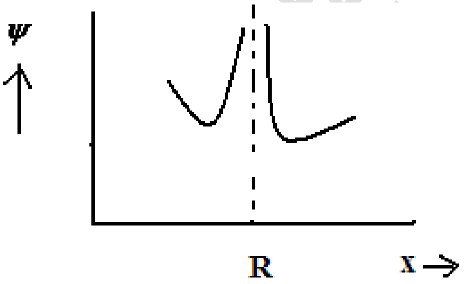
Physically acceptable wave function Ψ must satisfy the following conditions,

1. Ψ is single valued everywhere



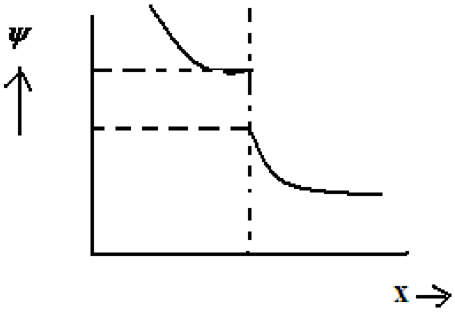
If Ψ has more than one value at any point (at P, Ψ has f_1, f_2 and f_3) it would mean more than one value of probability of finding the particle at that point which is obviously ridiculous. Therefore, Ψ must be single valued everywhere.

2. Ψ is finite everywhere



If Ψ is infinite at a point R there will be large probability of finding the particle at that point. This violates the uncertainty principle, therefore Ψ must have a finite or zero value at that point.

3. Ψ and its first derivatives dΨ/dx with respect to its variables are continuous everywhere



Ψ and its first derivatives must be continuous. Since the probability can have any value between zero and one, the wave function must be continuous.

Schrodinger's Time Independent One Dimensional Wave Equation

Based on de-Broglie idea of matter waves, Schrödinger developed a mathematical theory for a particle of mass 'm' moving with a velocity 'v' along x-direction associated with a wave of wavelength,

$$\lambda = \frac{h}{mv} = \frac{h}{p}$$

Where, p = mv is the momentum of the particle.

Let a wave function Ψ describing the de-Broglie wave travelling in +ve x-direction is given by,

$$\psi = Ae^{i(kx-\omega t)} \rightarrow (1)$$

Where Ψ is a total wave function, A is a constant and ω is angular frequency of wave.

Let us differentiate Ψ (in equation 1) twice with respect to 'x' then

$$\frac{d\psi}{dx} = A(ik)e^{i(kx-\omega t)}$$

$$\frac{d^2\psi}{dx^2} = A(ik)^2 e^{i(kx-\omega t)}$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi \qquad or \qquad \frac{d^2\psi}{dx^2} + k^2\psi = 0 \qquad \rightarrow (2) \quad \because i^2 = -1$$

$$But \qquad k = \frac{2\pi}{\lambda} \qquad and \qquad \lambda = \frac{h}{mv}$$

$$\therefore k = \frac{2\pi mv}{h} \quad \text{or} \quad k^2 = \frac{4\pi^2 m^2 v^2}{h^2}$$

Hence equation (2) becomes,

$$\frac{d^2\psi}{dx^2} + \frac{4\pi^2 m^2 v^2}{h^2}\psi = 0 \qquad \rightarrow (3)$$

The total energy E of the particle is the sum of kinetic energy T and potential energy V,

$$\therefore E = T + V$$

$$\text{But} \quad T = \frac{1}{2}mv^2 \qquad \therefore \frac{1}{2}mv^2 = (E - V)$$

$$\text{or} \quad mv^2 = 2(\text{E} - \text{V}) \qquad \rightarrow (4)$$

Substitute this value of mv^2 in equation (3) we get

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0 \quad \rightarrow (5)$$

This is known as time independent 1 - dimensional Schrödinger equation.

Equation (5) can also be extended for 3-dimensional space as,

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} + \frac{d^2\psi}{dz^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0 \rightarrow (6)$$

$$or \qquad \nabla^2\psi + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0 \quad \rightarrow (7)$$

$$where \qquad \nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$$

Equation (6) and (7) are the 3-dimensional time independent Schrödinger wave equation, where Ψ is $\Psi(x, y, z)$.

Eigen Functions and Eigen Values

"Eigen functions are those wave functions of quantum mechanics which possess the properties that they are single valued, finite everywhere and also their first derivatives with respect to their variables are continuous everywhere".

When the Eigen functions are operated by quantum mechanical operators on physical quantities like momentum, energy etc., of a system, the possible values are observed and these values are called Eigen values".

Ex: 1. If an operator say d/dx operates on a wave function $\Psi = e^{ax}$, then

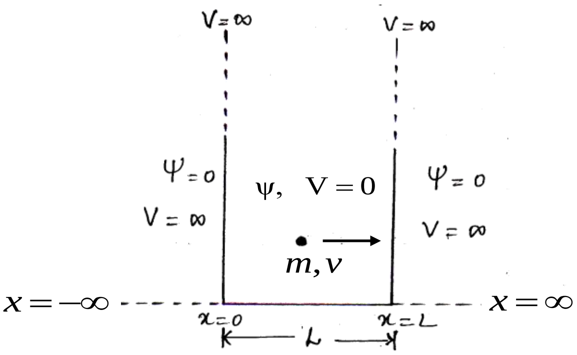
$$\frac{de^{ax}}{dx} = ae^{ax} = a\Psi$$

That is it produces the wave function multiplied by a constant. Such values obtained for a physical observable are called Eigen values.

Here 'a' is the Eigen value & $\Psi = e^{ax}$ is the Eigen function.

Applications of Schrödinger wave equation to particle trapped in a one dimensional square potential well

*** (Derivation of energy Eigen values and Eigen functions) ***



Consider a particle of mass ‘m’ moving with a speed ‘v’ along x-axis is confined to a box of length ‘L’ with perfectly rigid walls at x = 0 & x = L as shown in the figure.

The particle does not lose energy when it collides with the walls so that its total energy remains constant. The potential energy V of the particle is constant within the box which can be taken to be zero for convenience.

$$\therefore V = 0 \quad for \quad 0 < x < L \quad \rightarrow (1)$$

The potential energy of the particle is infinite on and beyond the walls of the box.

$$\therefore V = \infty \quad for \quad x \le 0 \;\; and \;\; x \ge L \rightarrow (2)$$

As the particle does not exist on the walls and beyond them, the wave function Ψ is zero.

$$\therefore \psi = 0 \quad for \;\; x \le 0 \;\; x \ge L \;\; \rightarrow (3)$$

The wave function Ψ exists within the box only.

∴ The Schrödinger’s time independent wave equation is,

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0 \qquad \rightarrow (4)$$

For the particle exists inside the box, V = 0

∴ Equation (4) becomes

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}E\psi = 0 \quad \rightarrow (5)$$

$$\text{let} \quad \frac{8\pi^2 mE}{h^2} = k^2 \quad \rightarrow (6)$$

∴Equation (5) becomes

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \quad \rightarrow (7)$$

This is the second order differential equation. The general solution of this equation is given by,

$$\psi = A\sin kx + B\cos kx \quad \rightarrow (8)$$

where, A & B are arbitrary constants, which are to be evaluated by using boundary conditions.

From the first boundary conditions, Ψ = 0 at x = 0, ∴Equation (8) becomes,

$$0 = A\sin 0 + B\cos 0$$

$$\text{Since,} \quad \sin 0 = 0 \quad \& \quad \cos 0 = 1, \quad \text{we have} \quad B = 0$$

$\therefore$ Equation (8) becomes,

$$\psi = A\sin kx \quad \rightarrow (9)$$

From second boundary conditions, $\Psi = 0$ at $x = L$, $\therefore$ Equation (9) becomes,

$$0 = A\sin kL$$

OR

$$A\sin kL = 0$$

$$\because A \neq 0,\ \sin kl = 0 \quad \text{for all values of} \quad kL = n\pi \quad \text{where } n = 1, 2, 3, \ldots\ldots.$$

$$\therefore k = \frac{n\pi}{L} \quad \rightarrow (10)$$

By substituting the value of k in equation (9) we get general wave function called Eigen wave function and Eigen energy equation.

$$\psi_n = A\sin\left(\frac{n\pi}{L}\right)x \rightarrow (11)$$

This is known as Eigen function or Eigen wave function.
Similarly by substituting the value of k in equation (6) we get,

$$\frac{8\pi^2 mE}{h^2} = \frac{n^2\pi^2}{L^2} \quad \text{or} \quad E = \frac{n^2h^2}{8mL^2}$$

$$\text{In general} \quad E_n = \frac{n^2h^2}{8mL^2} \quad \rightarrow (12)$$

This is the expression for Eigen values or Eigen energy values.

Thus, we see that in a potential well the particle cannot have an arbitrary energy, but it can have only discrete energy values corresponding to n = 1, 2, 3 ... are the Eigen values.

According to equation (11) if n = 0, $\Psi_n = 0$, which means that the particle doesn't present inside the box, which is not true. $\therefore$ The value of $E_n = 0$ for n = 0 is not acceptable. Hence the lowest allowed energy corresponding to n = 1 is called the 'zero-point energy or ground state energy'. Thus zero-point or ground state of energy of the particle in an infinite potential well is given by,

$$E_1 = \frac{h^2}{8mL^2}$$

The energy states corresponding to n >1 are called excited states.

Normalization:

To evaluate A in Eigen function Ψ_n, one has to perform the normalization of the wave function.

The allowed solutions of the Schrödinger equation are the Eigen functions, according to the equation.

$$\psi_n = A\sin\left(\frac{n\pi}{L}\right)x \quad \rightarrow (13)$$

The complex conjugate of equation (13) is,

$$\psi_n^* = A\sin\left(\frac{n\pi}{L}\right)x \quad \rightarrow (14)$$

To find the value of A, we use the normalization condition.

$$i.e., \quad \int_{-\infty}^{\infty} \psi_n^* \psi_n dx = \int_{-\infty}^{\infty} |\psi|^2 dx = 1$$

In this case, the particle exists only within the box of length (L). $\therefore$ The above equation can be written as,

$$\int_0^L |\psi_n|^2 dx = 1$$

By substituting the values of Ψ_n and Ψ_n^* in the above equation, we get

$$\int_0^L A^2 \sin^2 \frac{n\pi}{L} x \, dx = 1$$

$$A^2 \int_0^L \frac{1}{2}\left(1 - \cos\frac{2n\pi}{L}\right) x \, dx = 1 \qquad \left[\because \sin^2 A = \frac{1-\cos 2A}{2}\right]$$

$$\frac{A^2}{2}\left[\int_0^L dx - \int_0^L \cos\frac{2n\pi}{L} x dx\right] = 1$$

$$\frac{A^2}{2}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi}{L}\right)x\right]_0^L = 1$$

$$\frac{A^2}{2}\left[L - \frac{L}{2n\pi}\sin(2n\pi) - 0 + 0\right] = 1 \quad \text{Here, for any value of n, } \sin 2n\pi = 0$$

$$\therefore \frac{A^2 L}{2} = 1 \quad \text{or} \quad A^2 = \frac{2}{L} \quad \text{or} \quad A = \sqrt{\frac{2}{L}}$$

Thus, by substituting the value of A in equation (13) we get normalized wave functions or Eigen function of a particle in one dimensional infinite potential well.

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}\right)x \quad \rightarrow (15)$$

The first three eigen functions Ψ_1, Ψ_2, Ψ_3 together with the probability densities $|\psi_1|^2, |\psi_2|^2, |\psi_3|^2$ and eigen values E_1, E_2, E_3 are as shown in figure (a), (b) & (c) respectively for n = 1, 2 & 3.

For n = 1, this is the ground state and the particle is normally found in this state.

$$\text{Eigen function} \quad \psi_1 = A\sin\left(\frac{\pi}{L}\right)x$$

Ψ_1= 0 for both x = 0 and x = L and Ψ_1 has maximum value A for x = L/2

At x = 0 and x = L, $|\Psi_1|^2 = 0$ it means that the particle does not exist at the walls.

$|\Psi_1|^2$ is maximum at x = L/2, it means that the particle exist at the centre of the well.

$$E_1 = \frac{h^2}{8mL^2}$$

This is the energy eigen energy value for ground sate.

For first excited state, n = 2

$$\therefore \quad \psi_2 = A\sin\left(\frac{2\pi}{L}\right)x$$

Ψ_2 = 0 for x = 0, L/2 and L and Ψ_2 reaches maximum value for x = L/4 and 3L/4.

At x = 0, L/2 and L, $|\Psi_2|^2 = 0$ it means that the particle does not exist at 0, L/2 and L.

$|\Psi_2|^2$ is maximum at x = L/4 and 3L/4

Energy Eigen values can be calculated by using equation,

$$E_2 = \frac{4h^2}{8mL^2} \quad \text{or} \quad E_2 = 4E_1$$

This is the equation to calculate the energy of the particle in first excited state.

For second excited state, n = 3

$$\psi_3 = A\sin\left(\frac{3\pi}{L}\right)x$$

$\Psi_3 = 0$ for x = 0, L/3, 2L/3 and L, and Ψ_3 reaches maximum value for x = L/6, L/2 and 5L/6.
At x = 0, L/3, 2L/3 and L, $|\Psi_3|^2 = 0$ it means that the particle does not exist at 0, L/3, 2L/3 and L.
$|\Psi_3|^2$ is maximum at x = L/6, L/2 and 5L/6.
Energy Eigen values can be calculated by using equation,

$$E_3 = \frac{9h^2}{8mL^2} \quad \text{or} \quad E_3 = 9E_1$$

This is the equation to calculate the energy of the particle in second excited state.

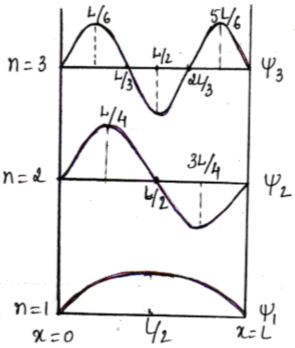


Fig (a)
Wave Functions

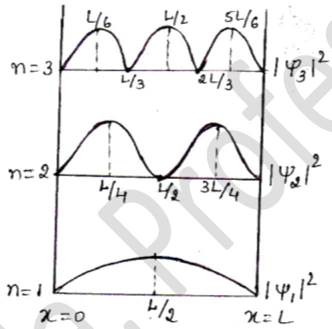


Fig (b)
Probability Functions

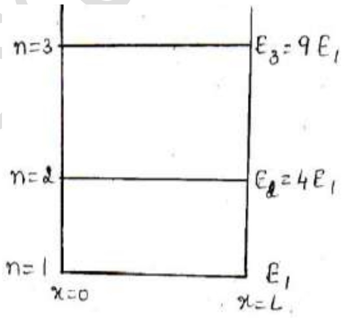


Fig (c)
Energy Levels.

Problems:

1. Calculate the zero point energy for an electron in a box of width 10Å.

$$Given,\ L = 10\overset{0}{A} = 10\text{x}10^{-10}\text{m},\ E_1 \text{ or } E_0 = ?$$

$$w.k.t; \quad E_n = \frac{n^2h^2}{8mL^2}$$

for ground state n = 1

$$E_1 = \frac{h^2}{8mL^2} = \frac{\left(6.63\times10^{-34}\right)^2}{8\times9.1\times10^{-31}\times\left(10\times10^{-10}\right)^2} = 6.038\times10^{-20}\,J$$

$$E_1 = \frac{6.038\times10^{-20}}{1.602\times10^{-19}}eV = 0.376\,eV$$

2. An electron is bound in a one dimensional potential well of width 1Å, but of infinite height. Find the energy value for the electron in the ground state.

$$Given,\ L = 1\overset{0}{A} = 1\text{x}10^{-10}\text{m},\quad E_1 \text{ or } E_0 = ?$$

$$w.k.t; \quad E_n = \frac{n^2h^2}{8mL^2}$$

for ground state n = 1

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.63\times10^{-34})^2}{8\times9.1\times10^{-31}\times(1\times10^{-10})^2} = 6.038\times10^{-18}\,J$$

$$E_1 = \frac{6.038\times10^{-18}}{1.602\times10^{-19}}eV = 37.69\,eV$$

3. An electron is trapped in a one-dimensional box of length 0.1 nm. Calculate the energy required to excite the electron from its ground state to the 2nd excited state.

Given, $L = 0.1\,\text{nm} = 0.1\text{x}10^{-9}\,\text{m}$, E_1 or E_0 = ?

$$w.k.t; \quad E_n = \frac{n^2h^2}{8mL^2}$$

for ground state n = 1

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.63\times10^{-34})^2}{8\times9.1\times10^{-31}\times(0.1\times10^{-9})^2} = 6.038\times10^{-18}\,J$$

$$E_1 = \frac{6.038\times10^{-18}}{1.602\times10^{-19}}eV = 37.691\,eV$$

for 2nd excited state n = 3

$$E_3 = \frac{9h^2}{8mL^2} = \frac{9\times(6.63\times10^{-34})^2}{8\times9.1\times10^{-31}\times(0.1\times10^{-9})^2} = 339.219eV$$

$$E_3 = 9\times E_1 = 9\times37.691 = 339.219\,eV$$

The energy required to excite the electron from its ground state to the 2nd excited state is,

$$\therefore E = E_3 - E_1 = (339.219 - 37.691)eV$$

$$E = 301.528\,eV$$

4. Calculate the lowest energy of the system consisting of three electrons in a one - dimensional potential box of length 1Å.

Given, $L = 1\overset{0}{\text{A}} = 1\text{x}10^{-10}\,\text{m}$, E_1 or E_0 = ?

$$w.k.t; \quad E_n = \frac{n^2h^2}{8mL^2}$$

for lowest energy n = 1

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.63\times10^{-34})^2}{8\times9.1\times10^{-31}\times(1\times10^{-10})^2} = 1.8114\times10^{-17}\,J$$

$$E_1 = \frac{1.8114\times10^{-17}}{1.602\times10^{-19}}eV = 113.07\,eV$$

5. An electron is constrained to a one-dimensional box of side 1nm. Calculate the first 3-eigen values in electron volt.

Given, $L = 1\,\text{nm} = 1\text{x}10^{-9}\,\text{m}$, E_1 or $E_0 = ?$, $E_2 = ?$, $E_3 = ?$

$$w.k.t;\quad E_n = \frac{n^2h^2}{8mL^2}, \qquad \text{for ground state n} = 1$$

$$E_1 = \frac{h^2}{8mL^2} = \frac{\left(6.63\times10^{-34}\right)^2}{8\times9.1\times10^{-31}\times\left(1\times10^{-9}\right)^2} = 6.038\times10^{-20}J$$

$$E_1 = \frac{6.038\times10^{-20}}{1.602\times10^{-19}}eV = 0.376eV$$

$$\text{For 2nd \& 3rd excited state} \quad \text{n} = 2\ \&\ \text{n} = 3$$

$$E_2 = 4\times E_1 = 4\times0.376 = 1.504eV$$

$$E_3 = 9\times E_1 = 9\times0.376 = 3.384eV$$

6. An electron is trapped in one-dimensional infinite potential box of width 0.1nm. Calculate its wavelengths and energies corresponding to first two excited states.

Given, $L = 0.1\,\text{nm} = 0.1\text{x}10^{-9}\,\text{m}$, $E_2 = ?$, $E_3 = ?$, $\lambda_2 = ?$, $\lambda_3 = ?$

$$w.k.t;\quad E_n = \frac{n^2h^2}{8mL^2}$$

$$\text{for first excited state n} = 2$$

$$E_2 = \frac{4h^2}{8mL^2} = \frac{4\left(6.63\times10^{-34}\right)^2}{8\times9.1\times10^{-31}\times\left(0.1\times10^{-9}\right)^2} = 2.415\times10^{-17}J$$

$$E_2 = \frac{2.415\times10^{-17}}{1.602\times10^{-19}}eV = 150.76eV$$

$$\text{for second excited state n} = 3$$

$$E_3 = \frac{9h^2}{8mL^2} = \frac{9\left(6.63\times10^{-34}\right)^2}{8\times9.1\times10^{-31}\times\left(0.1\times10^{-9}\right)^2} = 5.434\times10^{-17}J$$

$$E_3 = \frac{5.4354\times10^{-17}}{1.602\times10^{-19}}eV = 339.215eV$$

$$\text{w.k.t;}\qquad \lambda_n = \frac{2L}{n}$$

$$\text{For first excited state} \qquad \text{n} = 2$$

$$\lambda_2 = \frac{2L}{2}$$

$$\lambda_2 = L = 0.1\times10^{-9}m = 0.1nm$$

$$\text{For second excited state} \quad \text{n} = 3$$

$$\lambda_3 = \frac{2L}{3} = \frac{2\times0.1\times10^{-9}}{3}$$

$$\lambda_3 = 6.666\times10^{-11}m = 0.066\times10^{-9}m = 0.066\,nm$$

7. An electron is trapped in one-dimensional infinite potential box of width 0.15nm. Calculate the amount of energy required to excite an electron from ground state to 3rd excited state.

Given, $L = 0.15\,\text{nm} = 0.15\text{x}10^{-9}\,\text{m}$, $E = E_3 - E_2 = ?$

$$w.k.t; \quad E_n = \frac{n^2h^2}{8mL^2}$$

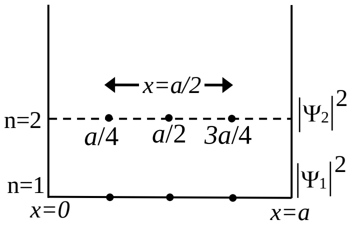
for ground state n = 1 and for third excited state n = 4

$$\therefore E = E_3 - E_2 = \frac{4^2h^2}{8mL^2} - \frac{h^2}{8mL^2} = \frac{(16-1)h^2}{8mL^2} = \frac{(15)x(6.626\times10^{-34})^2}{8\times9.1\times10^{-31}\times(0.15\times10^{-9})^2} = 9.102x10^{-19}$$

$$E = \frac{9.102x10^{-19}}{1.602\times10^{-19}}eV = 5.68eV$$

8. A quantum particle confined to one dimensional box of width ‘*a*’ in its first excited state. What is the probability of finding the particle over an interval of ‘*a*/2’ marked symmetrically at the centre of the box?

Soln.



The probability of finding the particle can be obtained by

$$p_n = \int_{-\infty}^{\infty} |\psi_n|^2 dx$$

Let a particle be in box of width ‘*a*’,. let p₂ be the probability occupation in the region (*a/2 = 3a/4 - a/4*) symmetrically at the centre therefore the probability of finding the particle in this region in first excited state is obtained by

$$p_2 = \int_{a/4}^{3a/4} |\psi_2|^2 dx = \int_{a/4}^{3a/4} \left|\sqrt{\tfrac{2}{a}} \sin\left(\tfrac{n\pi}{a}\right)x\right|^2 dx = \int_{a/4}^{3a/4} \tfrac{2}{a} \sin^2\left(\tfrac{2\pi}{a}\right)x dx$$

$$\therefore p_2 = \tfrac{2}{a}\int_{a/4}^{3a/4} \tfrac{1}{2}\left[1 - \cos2\left(\tfrac{2\pi}{a}\right)\right]x\, dx = \tfrac{1}{a}\left[x\Big|_{a/4}^{3a/4} - \tfrac{a}{2x2\pi}\sin2\left(\tfrac{2\pi}{a}\right)x\Big|_{a/4}^{3a/4}\right]$$

$$p_2 = \tfrac{1}{a}\left[3a/4 - a/4 + \tfrac{a}{2x2\pi}\sin2\left(\tfrac{2\pi}{a}\right)3a/4 - \tfrac{a}{2x2\pi}\sin2\left(\tfrac{2\pi}{a}\right)a/4\right] = \tfrac{1}{a}\left[a/2 + 0 - 0\right] = \tfrac{1}{2} = 0.5 = 50\%$$

The probability of occupation in the region *a*/2 at the centre of the box in the 1st excited state is 50%.

9. The ground state energy of an electron in an infinite well is 5.6 MeV. If the width of the well is doubled, calculate the ground state energy.

Given data: $E_1 = 5.6\, meV = 5.6\times10^{-3}eV \; when\; width\; a, \; E_1^I = ?\, when\; width\; a^I = 2a,$

$$E_1 = \frac{h^2}{8ma^2} \;\&\; E_1^I = \frac{h^2}{8m(2a)^2}$$

$$\frac{E_1^I}{E_1} = \frac{\left(1/2\right)^2}{1} = \frac{1}{4}$$

$$\therefore E_1^I = \frac{E_1}{4} = \frac{5.6\times10^{-3}}{4} = 1.4\times10^{-3}eV = 1.4\, meV$$

10. An electron is trapped in a 1-D potential well of infinite height and of width of 0.1nm. Calculate the energy required to excite it from its ground state to fifth excited state.

Given data: a = 0.1nm = 0.1x10^{-9}m, E = E_6-E_1 = ?

$E = \frac{n^2h^2}{8ma^2}$ (Note: when n=1 ground state, n-6 for 5th first excited state)

$$Ground\ \ state\quad E_1 = \frac{1^2 \times \left(6.63\times10^{-34}\right)^2}{8\times9.1\times10^{-31}\times\left(0.1\times10^{-9}\right)^2} = 6.038\times10^{-18}\,J = 37.69eV$$

$$For\ 5th\ excited\ state,\ n = 6$$

$$\therefore E_6 = \frac{6^2 \times \left(6.63\times10^{-34}\right)^2}{8\times9.1\times10^{-31}\times\left(0.1\times10^{-9}\right)^2} = 2.174\times10^{-16}\,J = 1357eV$$

$$\therefore E = E_6 - E_1 = 1357eV - 37.69eV = 1319.31eV$$

Questions:

1. Define wave function? Mention its basic properties.
2. What is a wave function? Explain its physical significance.
3. Discuss the physical interpretation of wave function?
4. Discuss about probability and normalisation condition.
5. Set up time independent Schrödinger's wave equation.
6. Derive Time independent Schrodinger wave equation for a particle moving in one dimension.
7. Explain the significance of wave function and set up time independent Schrödinger's wave equation.
8. Describe Eigen functions and Eigen values.
9. Derive the expressions for eigen values and eigen functions of a particle in one dimensional potential box.
10. Solve Schrödinger wave equation for allowed energy values in case of a particle in a potential box.
11. Obtain the expression for normalised wave function for a particle in one dimensional potential box.
12. Assuming the time independent Schrödinger's wave equation, discuss the solution for a particle in one dimensional potential well of infinite height,
13. Discuss and mention the energy eigen values, eigen wave functions and probability densities for a particle in 1-dimensional box for atleast 3 states.
14. Discuss the energies of a free particle using time independent Schrodinger equation.

Problems:

1. An electron is confined to a one dimensional box of width 1 nm. Calculate the first three Eigen values in eV.
2. An electron is bound in a one dimensional box of width 4 X 10^{-10} m. compute the energy and de-Broglie wavelength of ground and first excited states.
3. Find the energy of an electron in the ground state, when it is trapped in an infinite potential well of width 2Å.
4. An electron is bound in one dimensional potential well of width 1Å but of infinite height. Find its energy values in ground state and first two exited states.
5. Estimate the time spent by an atom in the excited state during the excitation and de-excitation processes, when a spectral line of wavelength 546 nm and width 10^{-14} m is emitted.
6. An electron is confined to a box of length 10^{-9} m, calculate the minimum uncertainty in its velocity.
7. The position and momentum of 1 keV electron are simultaneously determined. If it position is located within 1Å, find the uncertainty in the determination of its momentum.
8. A spectral line of wavelength 4000 Å has a width of 8×10^{-5}Å. Evaluate the minimum time spent by the electrons in the upper energy state between the excitation and de-excitation processes.
9. The inherent uncertainty in the measurement of time spent by Iridium 191 nuclei in the excited state is found to be 1.4×10^{-10}s. Estimate the uncertainty that results in its energy in eV in the excited state.
10. An electron is bound in one dimensional potential well of width 0.18 nm. Find the energy value in eV of the second excited state.
11. The first excited state energy of an electron in an infinite well is 240 eV. What will be its ground state energy when the width of the potential well is doubled?
12. A quantum particle confined to one–dimensional box of width *‘a’* is in its first excited state. What is the probability of finding the particle over an interval of *a*/2 marked symmetrically at the center of the box.

Free Particle

Energy Eigen values for a free particle:

Free particle means, it is not under the influence of any kind of field or force. Thus it has zero potential, i.e., V = 0.

Hence Schrödinger's equation becomes,

$$\frac{d^2\psi}{dx^2} = -\frac{8\pi^2 m}{h^2}(E-V)\psi \ = 0$$

Since V = 0, $$\frac{d^2\psi}{dx^2} = -\frac{8\pi^2 m}{h^2}E\psi \ = 0$$

The above equation holds good for a particle for which the potential v=0 over the entire space (No boundaries at all).

We know that in the case of particle in an infinite potential well, the condition V=0 holds good only over a infinite width 'L' and outside region, V=∞,

Since for the free particle, V = 0 holds good everywhere, we can extend the case of particle in an infinite potential well to the free particle's case, by treating the width of the well to be infinity, i.e., by allowing L = ∞,

We have the equation for energy Eigen values for a particle in an infinite potential well as,

$$E = \frac{n^2h^2}{8mL^2}$$

Where, n = 1, 2, 3, ...

Rearranging the above equation, we have,

$$n = \frac{2L}{h}\sqrt{2Em}$$

Here, we see that for a particle with constant energy E but confined in the well, n depends mainly on 'L'. Hence as L→ ∞, n→∞. If the particle is no more confined in any sort of well but free, at that time it also follows that n = ∞, which essentially means that a free particle can have any energy i.e., the energy Eigen values or the possible values of energy are infinite in number. Keeping the energy level representation in the mind, we say that the permitted energy values are continues. All these mean, there is no discreteness in the allowed energy values. In other word, there is no quantization of energy in case of a free particle and the problem is dealt in classical mechanics. Thus a free particle is a classical entity.

https://www.youtube.com/watch?v=tlM9vq-bepA

https://www.youtube.com/watch?v=v9DPzMoWpc0

https://www.youtube.com/watch?v=8l4x4vbMP0c

MYSURU ROYAL INSTITUTE OF TECHNOLOGY

DEPARTMENT OF PHYSICS

Applied Physics for EEE Stream : 2023-24

Module – 2 :

Electrical Properties of Solids

Syllabus:

Electrical Properties of Solids **8 hrs**

Conductors : Quantum Free Electron Theory of Metals: Assumptions, Fermi-energy, Fermi factor, Variation of Fermi Factor with Temperature and Energy, Mention of expression for electrical conductivity.

Dielectric Properties: Polar and non-polar dielectrics, Electrical Polarization Mechanisms, internal fields in solids, Clausius-Mossotti equation (Derivation), Solid, Liquid and Gaseous dielectrics. Application of dielectrics in transformers, Capacitors, Electrical Insulation. Numerical Problems.

Superconductivity: Introduction to Superconductors, Temperature dependence of resistivity, Meissner Effect, Critical Field, Temperature dependence of Critical field, Types of Super Conductors, BCS theory (Qualitative), High Temperature superconductivity, SQUID, MAGLEV, Numerical problems.

Pre requisites: Electron concept, Basics of Electrical conductivity

Self-learning: Classical free electron theory, Resistivity and Mobility

Electrical Conductivity in Metals

Introduction:

Materials can be classified into three types based on the conductivity of heat and electricity. They are;

1. Conductors (Example : Metals – Copper, Aluminum, Silver, Gold)
2. Semiconductors (Example : Germanium, Silicon)
3. Insulators (Example : Wood, Mica, Glass)

Electron Theory of Metals

The electron theory of metals explains the following:

- Structural, electrical and thermal properties of materials.
- Elasticity, cohesive force and binding in solids.
- Behaviour of conductors, semiconductors, insulators etc.,

In solids, electrons in the outermost orbit of atoms are called valance electrons, which determine the properties of the materials. The electron theory is applicable to all solids (both metals and non-metals). This theory explains the electrical, thermal and magnetic properties of solids.

Quantum Free Electron Theory (Summerfeld Theory):

To overcome the drawbacks of classical free electron theory, Sommerfeld proposed quantum free electron theory. He treated electron as a quantum particle. He retains the vital features of classical free electron theory and included the Pauli Exclusion Principle & Fermi-Dirac statistics. The following are the **assumptions** of quantum free electron theory.

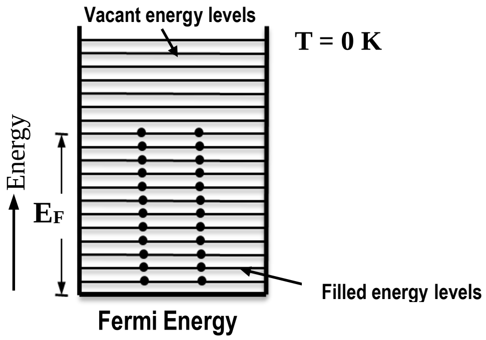
1. The free electrons in a metal can have only discrete energy values. Thus the energies are quantized.
2. The electrons obey Pauli's Exclusion Principle, which states that there cannot be more than two electrons in any energy level.
3. The distribution of electrons in various energy levels obeys the Fermi-Dirac quantum statistics.
4. Free electrons have the same potential energy everywhere within the metal, because the potential due to ionic cores is uniform throughout the metal.
5. The force of attraction between electrons & lattice ions and the force of repulsion between electrons can be neglected.
6. Electrons are treated as wave-like particles.

Fermi - level, Fermi - energy and Fermi - factor

As we know that for a metal containing N atoms, there will be N number of energy levels in each band. According to Pauli's exclusion principle, each energy level can accommodate a maximum of two electrons, one with spin up (+½) and the other with spin down (-½). At absolute zero temperature, two electrons with opposite spins will occupy the lowest available energy level. The next two electrons with opposite spins will occupy the next energy level and so on. Thus, the top most energy level occupied by electrons at absolute zero temperature is called **Fermi-energy level**. The energy corresponding to that energy level is called **Fermi-energy**.

The energy of the highest occupied level at zero-degree absolute is called Fermi energy, and the energy level is referred to as the Fermi level. The Fermi energy is denoted as **E_F**.

All energy levels below Fermi level are completely filled and above which all energy levels are completely empty.



At temperatures above absolute zero, the electrons get thermally excited and move up to higher energy levels. As a result, there will be many vacant energy levels below as well as above Fermi energy level. Under thermal equilibrium, the distribution of electrons among various energy levels is given by statistical function *f*(E). The function ***f*(E)** is called **Fermi-factor** and this gives the probability of occupation of a given energy level under thermal equilibrium. The expression for *f*(E) is given by

$$f(E) = \frac{1}{e^{(E-E_F)/kT} + 1}$$

Where ***f*(E)** is called Fermi-Dirac distribution function of Fermi factor, **E_F** is the Fermi energy, **k** is the Boltzmann constant and **T** is the temperature of metal under thermal equilibrium.

Note: 1. The Fermi-Dirac distribution *f*(E) is used to calculate the probability of an electron occupying a certain energy level.

2. The distribution of electrons among the different energy levels as a function of temperature is known as Fermi-Dirac distribution function.

Variation of Fermi factor with Energy and Temperature

Let us consider the different cases by considering the Fermi factor equation

$$f(E) = \frac{1}{e^{(E-E_F)/kT} + 1}$$

Case (i) : *f*(E) for E < E_F at T = 0 K;

When; E < E_F & T = 0 K, from the probability function *f*(E) we have

$$f(E) = \frac{1}{e^{-\infty} + 1} = \frac{1}{0+1} = 1$$

i.e., *f*(E) = 1 for E < E_F at T = 0 K.

This implies that at absolute zero temperature, all the energy levels below E_F are 100% occupied which is true from the definition of Fermi energy.

<u>Case (ii)</u> : *f*(E) for E > E_F at T = 0 K;

When E > E_F & T = 0 K, then *f*(E) becomes

$$f(E) = \frac{1}{e^{\infty}+1} = \frac{1}{\infty+1} = \frac{1}{\infty} = 0$$

i.e., *f*(E) = 0 for E > E_F at T = 0 K.

This implies that at absolute zero temperature, all the energy levels above E_F are unoccupied (completely empty) which is true from the definition of Fermi energy.

<u>Case (iii)</u> : *f*(E) for E = E_F at T = 0 K;

When E = E_F & T = 0 K, then *f*(E) becomes

$$f(E) = \frac{1}{e^{\%}+1} = \text{Indeterminate}$$

i.e., *f*(E) = ∞ for E = E_F at T = 0 K.

Hence, the occupation of Fermi level at T = 0 K has an undetermined value ranging between zero and unity (0 & 1). The Fermi-Dirac distribution function is discontinuous at E = E_F for T = 0 K.

<u>Case (iv)</u> :- *f*(E) for E = E_F at T > 0 K;

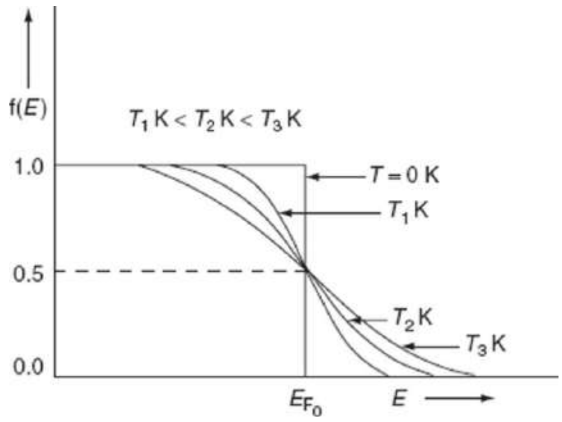
When E = E_F & T > 0 K, then *f*(E) becomes

$$f(E) = \frac{1}{e^{0}+1} = \frac{1}{1+1} = \frac{1}{2}$$

i.e., *f*(E) = ½ for E = E_F at T > 0 K.

If E « E_F, the probability starts decreasing from 1 and reaches 0.5 (½) at E = E_F and for E > E_F, it further falls off as shown in figure. In conclusion, the Fermi energy is the most probable or average energy of the electrons in a solid.

The variation of Fermi factor with energy and temperature is as shown in figure given below.



Importance of Fermi Energy

- Fermi energy level is used to separate the vacant and filled states at 0 K.
- It is used to know the status of the electrons.
- Electrons are completely filled below the Fermi energy level and completely empty above the Fermi level at 0 K.
- Above 0 K some electrons absorb thermal energy and they jump to the higher energy levels.

Expressions for electrical conductivity and resistivity as per quantum free electron theory.

$$\sigma = \frac{ne^2}{m^*}\left(\frac{\lambda}{v_F}\right) \qquad or \qquad \rho = \frac{1}{\sigma} = \frac{m^*}{ne^2}\left(\frac{v_F}{\lambda}\right)$$

Where σ is the electrical conductivity of the metal

n is the number of free electrons per unit volume

m* is the effective mass of an electron

λ is the mean free path of electron

v_F is the Fermi velocity of electron

ρ is the resistivity of the metal

Merits or Success of Quantum free electron theory:

The quantum free electron theory solves the flaws of the classical free electron theory which are discussed below.

1. **Specific heat of free electrons:** According to quantum free electron theory, the electrons occupying energy levels close to E_F can absorb heat energy. Such electrons constitute a very small percentage of the total number of free electrons. Hence the specific heat of free electrons is given by

$$C_V = \frac{2k}{E_F}RT$$

Since the value of E_F ranges from 1 to 10 eV, by taking a typical value of E_F = 5 eV, we get

$$\frac{2k}{E_F} \approx 10^{-4}$$

$$\therefore C_V = 10^{-4}RT$$

which is in agrees well with the experimental results.

2. **Temperature dependence of resistivity or conductivity in metals:** According to quantum free electron theory, the expressions for electrical conductivity & resistivity of a metal are given by

$$\sigma = \frac{ne^2}{m^*}\left(\frac{\lambda}{v_F}\right) \qquad or \qquad \rho = \frac{m^* v_F}{ne^2\lambda}\left(\frac{v_F}{\lambda}\right)$$

In the above expression only the mean free path λ is the temperature dependent quantity.

$$\therefore \sigma \propto \lambda \quad \rightarrow \quad (1)$$

In classical theory, the collision was seen as a particle bouncing off another. In the quantum understanding, an electron is viewed as a wave travelling through the medium. If **r** represents the amplitude of the oscillation of the lattice ions can be considered to present a circular cross section of area πr^2 that blocks the path of the electron waves. Hence electron waves are scattered more effectively results in a reduction of mean free path (**λ**)**.** Thus **λ** is inversely proportional to the area of cross section.

$$\text{i.e.,} \quad \lambda \propto \frac{1}{\pi r^2} \quad \rightarrow \quad (2)$$

But the area of cross section πr^2 is directly proportional to the absolute temperature.

$$i.e., \quad \pi r^2 \propto T$$

$$\therefore \quad \lambda \propto \frac{1}{T} \quad \rightarrow \quad (3)$$

By substituting the values of λ in equation (1) we get

$$\sigma \propto \frac{1}{T} \quad \text{or} \quad \rho \propto T$$

This is exactly same as the experimental prediction. Thus quantum free electron theory properly explains the dependence of σ on T.

3. **Dependence of electrical conductivity on electron concentration:** According to quantum free electro theory, the electrical conductivity in metals is given by

$$\sigma = \frac{ne^2}{m^*}\left(\frac{\lambda}{v_F}\right)$$

From the above equation it is clear that the electrical conductivity depends on both the electron concentration n and $\left(\frac{\lambda}{v_F}\right)$.

If we compare the cases of copper and aluminium, the value of n for aluminium is 2.13 times higher than that of copper. But the value of **λ/v_F** for copper is about 3.73 times higher than that of aluminium. Thus the conductivity of copper is more than that of aluminium.

Problems :

1. The free electron density of aluminium is 18.10x10^{28} m^{-3}. Calculate its Fermi energy at 0 K. Planck's constant and mass of free electron are 6.626x10^{-34} Js and 9.1x10^{-31} kg.

Solns.
n = 18.10x10^{28} m^{-3}
Planck's constant, h = 6.626x10^{-34} Js
Mass of an electron, m = 9.11x10^{-31} kg

$$E_{F_0} = \left(\frac{h^2}{8m}\right)\left(\frac{3n}{\pi}\right)^{2/3} = \left(\frac{(60626x10^{-34})^2}{8x9.11x10^{-31}}\right)\left(\frac{3x18.10x10^{28}}{\pi}\right)^{2/3}$$

$$= 1.8689x10^{-18}\,J = \frac{1.8689x10^{-18}}{1.602x10^{-19}} = 11.68 \quad eV$$

2. Calculate the density of states for copper at the Fermi level for T = 0 K. Given that, electron density of copper is $8.5x10^{28}$ electrons $/m^3$.

Solns.

n = $8.5x10^{28}\ m^{-3}$

Planck's constant, h = $6.626x10^{-34}$ Js

Mass of an electron, m = $9.1x10^{-31}$ kg

$$g(E)dE = \frac{\pi}{2}\left(\frac{8m}{h^2}\right)^{3/2} E_F^{\frac{1}{2}} dE \quad where, \quad E_F = \left(\frac{h^2}{8m}\right)\left(\frac{3n}{\pi}\right)^{2/3}$$

$$E_F = \left(\frac{h^2}{8m}\right)\left(\frac{3n}{\pi}\right)^{2/3} = \left(\frac{\left(6.626x10^{-34}\right)^2}{8x9.1x10^{-31}}\right)\left(\frac{3x8.5x10^{28}}{\pi}\right)^{2/3} = 1.1293x10^{-18} J = 7.05\ eV$$

$$g(E) = \frac{\pi}{2}\left(\frac{8m}{h^2}\right)^{3/2} E_F^{\frac{1}{2}} = \frac{\pi}{2}\left(\frac{8x9.1x10^{-31}}{\left(6.626x10^{-34}\right)^2}\right)^{3/2} x\ 7.05^{1/2}$$

3. Find the probability of an electron occupying an energy level 0.02 eV above the Fermi level at 200 K and 400 K in a material.

Solns.

$E - E_F = 0.02\ eV = 0.02x1.602x10^{-19}\ J = 3.204\ x10^{-21}\ J$

$T_1 = 200$ K & $T_2 = 400$ K

$$f(E)_{atT_1} = \frac{1}{e^{(E-E_F)/kT_1}+1} == \frac{1}{e^{\left(\frac{3.204x10^{-21}}{1.38x10^{-23}x200}\right)}+1} = \frac{1}{e^{(1.1594)}+1} = \frac{1}{3.188+1} = 0.24$$

$$f(E)_{atT_2} = \frac{1}{e^{(E-E_F)/kT_2}+1} == \frac{1}{e^{\left(\frac{3.204x10^{-21}}{1.38x10^{-23}x400}\right)}+1} = \frac{1}{e^{(0.5797)}+1} = \frac{1}{1.7855+1} = 0.36$$

4. Show that the sum of the probability of occupancy of an energy state at ΔE above the Fermi level and that at ΔE below the Fermi level is unity.

Solns.

Let $f(E)_E^a$ is the probability of ocuupation abobe the Fermi level at energy $E = (E_F + \Delta E)$

&

$f(E)_E^b$ is the probability of ocuupation below the Fermi level at $E = (E_F - \Delta E)$

$S.T.$ $\quad f(E)_E^a + f(E)_E^b = 1$

We know that $f(E) = \dfrac{1}{e^{(E-E_F)/kT}+1}$

For energy above the Fermi level, $\mathrm{E} = (\mathrm{E_F} + \Delta\mathrm{E})$ or $\mathrm{E} - \mathrm{E_F} = \Delta\mathrm{E}$ $\therefore$ $f(E)_E^a = \dfrac{1}{e^{(\Delta E)/kT}+1}$

For energy below the Fermi level, $\mathrm{E} = (\mathrm{E_F} - \Delta\mathrm{E})$ or $\mathrm{E} - \mathrm{E_F} = -\Delta\mathrm{E}$ $\therefore$ $f(E)_E^b = \dfrac{1}{e^{-(\Delta E)/kT}+1}$

$$\therefore f(E)_E^a + f(E)_E^b = \frac{1}{e^{(\Delta E)/kT}+1} + \frac{1}{e^{-(\Delta E)/kT}+1}$$

Put $e^{(\Delta E)/kT} = x$, $\therefore$ $e^{-(\Delta E)/kT} = \dfrac{1}{x}$, Hence.,

$$f(E)_E^a + f(E)_E^b = \frac{1}{x+1} + \frac{1}{\frac{1}{x}+1} = \frac{1}{x+1} + \frac{1}{\frac{1+x}{x}} = \frac{1}{x+1} + \frac{x}{1+x} = \frac{1+x}{x+1} = 1$$

$$\therefore \quad f(E)_E^a + f(E)_E^b = 1$$

5. Calculate the probability of an electron occupying an energy level of 0.05 eV at 200 K above and below the Fermi level.

Soln.

$\mathrm{E} - \mathrm{E_F} = 0.05\,\mathrm{eV}$, $\mathrm{T} = 200\,\mathrm{K}$

$f(E)_E^a = ?$ & $f(E)_E^b = ?$

For energy above the Fermi level, $\mathrm{E} = (\mathrm{E_F} + \Delta\mathrm{E})$ or $\mathrm{E} - \mathrm{E_F} = 0.05\,\mathrm{eV} = 0.05\mathrm{x}1.602\mathrm{x}10^{-19}\,\mathrm{J}$

$$\therefore \text{ At, T} = 200\text{ K.,} \quad f(E)_E^a = \frac{1}{e^{\left(\frac{0.05\text{x}1.602\text{x}10^{-19}}{1.38\text{x}10^{-23}\text{x}500}\right)}+1} = \frac{1}{e^{1.16087}+1} = \frac{1}{3.1927+1} = \frac{1}{4.1927} = 0.2385$$

For energy below the Fermi level, $\mathrm{E} = (\mathrm{E_F} - \Delta\mathrm{E})$ or $\mathrm{E} - \mathrm{E_F} = -\Delta\mathrm{E}$ $\therefore$ $f(E)_E^b = \dfrac{1}{e^{-(\Delta E)/kT}+1}$

$$\therefore \text{ At, T} = 200\text{ K.,} \quad f(E)_E^b = \frac{1}{e^{-\left(\frac{0.05\text{x}1.602\text{x}10^{-19}}{1.38\text{x}10^{-23}\text{x}500}\right)}+1} = \frac{1}{e^{-1.16087}+1} = \frac{1}{0.3132+1} = \frac{1}{1.3132} = 0.7615$$

$$\therefore f(E)_E^a + f(E)_E^b = 0.2385 + 0.7615 = 1$$

6. Find the temperature at which there is 1 % probability that a state with 0.5 eV energy above the Fermi energy is occupied.

Soln.

$$\mathrm{E - E_F = 0.5\,eV = 0.5x1.602x10^{-19}\,J}$$

$$f(E) = 1\% = 0.01$$

$$\mathrm{T = ?}$$

$$W.K.T \qquad f(E) = \frac{1}{e^{(E-E_F)/kT}+1} \qquad or \qquad 0.01 = \frac{1}{e^{\left(\frac{0.5x1.602x10^{-19}}{1.38x10^{-23}xT}\right)}+1} = \frac{1}{e^{\left(\frac{5804}{T}\right)}+1}$$

$$or \qquad 0.01 = \frac{1}{e^{\left(\frac{0.5x1.602x10^{-19}}{1.38x10^{-23}xT}\right)}+1} = \frac{1}{e^{\left(\frac{5804}{T}\right)}+1}$$

$$\therefore \qquad e^{\left(\frac{5804}{T}\right)}+1 = \frac{1}{0.01} = 100 \quad \text{or} \quad e^{\left(\frac{5804}{T}\right)} = 100-1 = 99$$

$$\frac{5804}{T} = \ln(99) = 4.595 \quad \text{or} \quad \mathrm{T} = \frac{5804}{4.595} = 1263K$$

7. The Fermi level in potassium is 2.1 eV. What are the energies for which the probability of occupancy at 300 K are 0.99, 0.01 and 0.5?

Soln.

For potassium $\mathrm{E_F = 2.1\,eV}$

Find, $\mathrm{E_1 = \ ?}$ for $\mathrm{f(E_1) = 0.99}$

Find, $\mathrm{E_2 = \ ?}$ for $\mathrm{f(E_2) = 0.01}$

& Find, $\mathrm{E_3 = \ ?}$ for $\mathrm{f(E_3) = 0.5}$

$$W.K.T \qquad f(E) = \frac{1}{e^{(E-E_F)/kT}+1} \quad \text{or} \quad e^{(E-E_F)/kT} = \frac{1}{f(E)} - 1$$

Taking natural logarithm on both sides, we get,

$$\frac{E-E_F}{kT} = \ln\left(\frac{1}{f(E)}-1\right) \quad \text{or} \quad \mathrm{E = E_F + kT\ln}\left(\frac{1}{f(E)}-1\right)$$

$\therefore \mathrm{E_1}$ for $\mathrm{f(E_1)}$ is given by

$$\mathrm{E_1 = E_F + kT\ln}\left(\frac{1}{f(E_1)}-1\right) = 2.1 + \frac{1.38x10^{-23}x300}{1.602x10^{-19}}\ln\left(\frac{1}{0.99}-1\right)$$

$$= 2.1 + 0.02584\mathrm{x}(-4.5950) = 2.1 - 0.1187 = 1.9813\,eV$$

slly $\mathrm{E_2}$ for $\mathrm{f(E_2)}$ is given by

$$\mathrm{E_2 = E_F + kT\ln}\left(\frac{1}{f(E_2)}-1\right) = 2.1 + \frac{1.38x10^{-23}x300}{1.602x10^{-19}}\ln\left(\frac{1}{0.01}-1\right)$$

$$= 2.1 + 0.02584\mathrm{x}(4.5950) = 2.1 + 0.1187 = 2.2187\ eV$$

slly $\mathrm{E_3}$ for $\mathrm{f(E_3)}$ is given by

$$\mathrm{E_3 = E_F + kT\ln}\left(\frac{1}{f(E_3)}-1\right) = 2.1 + \frac{1.38x10^{-23}x300}{1.602x10^{-19}}\ln\left(\frac{1}{0.5}-1\right)$$

$$= 2.1 + 0.02584\mathrm{x}(0) = 2.1 + 0 = 2.1\,eV$$

8. Calculate the Fermi energy in eV for a metal at 0 K, whose density is 10500 kgm^{-3}, atomic weight is 107.9 and it has one conduction electron per atom.

Solns.

Density of metal, $\rho = 10500$ kgm^{-3}

Atomic weight of metal, wt. = 107.9

Fermi energy, E_F = ?

We know that the concentration of electrons in metal, n is given by

$$\mathrm{n} = \frac{\text{density}(\rho)\ \text{x Avogadro's costant}(\mathrm{N}_A)\ \text{x no. of free electrons per atom}}{\text{atomic weight (wt.)}}$$

$$= \frac{10500\ \text{x}\ 60022\text{x}10^{26}\ \text{x}\,1}{107.9} = 5.861x10^{28}m^{-3}$$

$$E_{F_0} = \left(\frac{h^2}{8m}\right)\left(\frac{3}{\pi}\right)^{2/3}(n)^{2/3} = \left(\frac{(60626x10^{-34})^2}{8x9.11x10^{-31}}\right)\left(\frac{3x5.816x10^{28}}{\pi}\right)^{2/3}$$

$$= 8.8173x10^{-18}J = \frac{8.8173x10^{-18}}{1.602x10^{-19}} = 5.51 \quad eV$$

REVIEW QUESTIONS

Long Answer Questions:

1. Define the terms: Fermi-level, Fermi-energy and Fermi-factor.
2. Mention the drawback/Failures of classical free electron theory and explain them.
3. What are the assumptions of quantum free electron theory?
4. Define the terms: Fermi-temperature, Fermi-velocity and density of states.
5. What are the successes of quantum free electron theory? Explain.
6. How quantum free electron theory successfully explain the failures of classical free electron theory.
7. What are the merits of quantum free electron theory? Explain.
8. Write down the Fermi-Dirac equation for the probability of occupation of an energy level E by an electron. Show that the probability of its occupancy by an electron is zero if E > E_F and unity if E < E_F at temperature 0 K.
9. Define Fermi factor. Explain Fermi Dirac distribution for electrons in a metal at temperature T = 0 K and T > 0 K.
10. Write an expression for the Fermi energy distribution function *f*(E) and discuss its behavior with change in temperature. Plot *f*(E) versus E for T = 0 K, and T > 0 K.
11. Explain the dependence of electrical conductivity on temperature and electron concentration on the basis of classical free electron theory.
12. Write down the difference between classical and quantum free electron theories.

Problems:

1. Obtain the value of *f*(E) for E - E_F = 0.01 eV at T = 300 K.
2. Find the probability that an energy level at 0.2 eV below Fermi level being occupied at temperatures 300 K and 900 K.
3. At what temperature can we expect a 10% probability that electron in silver have an energy which 1% above the Fermi energy? The Fermi energy of silver is 5.5 eV.
4. Evaluate the Fermi function for an energy 0.04 eV at T= 330 K above the Fermi energy.
5. Show that the probability of occupation above the fermi level is same as the non-occupation probability below the fermi level for given energy and temperature.

∗ ∗ ∗ ∗ ***END*** ∗ ∗ ∗ ∗

DIELECTRICS

Introduction:

Dielectrics are insulating or non-conducting materials and are used in many applications such as capacitors, memories, sensors and actuators. Dielectrics are insulating materials they exhibit the property of electrical polarization. In principle all dielectrics are insulators but all insulators are not dielectrics. Although these materials do not conduct electrical current when an electric field is applied, but the field may cause a slight shift in the balance of charge within the material to form an electric dipole.

1. Dielectrics are the substances which do not possess free electric charges under ordinary circumstances. e.g., glass, mica, plastic etc.,
2. Dielectric materials are those which have attaining polarization when electric field is applied.
3. A dielectric is mainly characterized by its dielectric constant.

Dielectric constant (ε_r or K)

The dielectric constant of a material is defined as the ratio of the permittivity of the medium (ε) to the permittivity of free space (ε_o). It can also be defined as the ratio of the capacitance of a capacitor with dielectric (C) to the capacitance of the same capacitor with air (C_0).

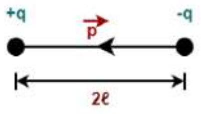
$$i.e., \quad \varepsilon_r = \frac{\varepsilon}{\varepsilon_o} \quad \text{or} \quad \varepsilon_r = \frac{C}{C_o}$$

$$i.e., \quad K = \frac{\varepsilon}{\varepsilon_o} \quad \text{or} \quad K = \frac{C}{C_o}$$

Where ε_r is called dielectric constant or relative permittivity (ability of the material to store electric charges).

Electric dipole and Dipole moment:

A pair of equal and opposite charges separated by a very small distance is called an electric dipole.



The product of the magnitude of either of the charges and the distance of their separation is called the dipole moment

$$i.e., \quad \mu = qx2l$$

Types of dielectrics:

Dielectrics are mainly divided into two groups, namely Polar and Non polar dielectrics.

Polar dielectrics: The molecules in which the center of gravity of positive charges is separated from the center of gravity of negative charges by a finite distance are called polar molecules. Polar dielectric materials possess permanent electric dipoles in the material and are oriented in random directions so that the net dipole moment of the material is zero in the absence of applied electric field. If polar dielectric materials are placed in the external electric field then all dipoles tend to align in the field direction and hence net dipole moment develops cross dielectric material.

Materials like KI, HCl, CO, H_2O, NH_3 etc., are the examples for polar dielectrics.

Non-Polar dielectrics: The molecule in which the center of gravity of positive charges (protons) coincides with center of gravity of negative charges (electrons), then the molecule is known as nonpolar molecules.

Non polar dielectric materials do not possess permanent electric dipoles. Thus the net dipole moment across the material is zero in the absence of external electric field. In non-polar dielectric materials dipoles are induced due to the applied electric field which results in the net dipole moment in the dielectric material in the direction of the applied field.

Elementary gasses like *He*, He, Ne, Ar & Xe and H_2, N_2, Cl_2 etc., are the examples for non-polar dielectrics.

Polarization: In the absence of external electric field, dipoles are randomly oriented within the dielectric and the net dipole moment is zero. When an electric field is applied to a dielectric slab, all the dipoles are oriented in the direction of the field and hence the dipole moment increases. This is known as polarization.
Polarization is defined as the net dipole moment per unit volume.

$$i.e., \qquad P = \frac{\mu}{\mathrm{v}}$$

Dielectric Polarization: The displacement of charges in the molecules of a dielectric under the action of applied electric field, leading to the development of dipole moment is called dielectric polarization or electrical polarization.
Polarization is directly proportional to the applied electric field and the number of molecules per unit volume.

$$i.e., P \propto E \quad \text{or} \quad P = \alpha E$$

$$and \quad p \propto NE \quad \text{or} \quad P = \alpha NE$$

Where α is known as polarizability. It is defined as the ratio of polarization to the applied electric field

$$i.e., \qquad \alpha = \frac{P}{\mathrm{E}}$$

We also know that, the flux density in the absence of dielectric is given by

$$D = \varepsilon_0 E$$

And the flux density in the presence of dielectric is given by

$$D = \varepsilon_0 \varepsilon_r E$$

By using the above equations, we can also show that,

$$p = \varepsilon_0 (\varepsilon_r - 1) E$$

Where P is polarization, ε_r is dielectric constant and $\varepsilon_0 = 8.854 \times 10^{-12} F/m$ is the permittivity of free space.

Types of Polarization: There are four types of polarization, namely

1. Electronic Polarization (P_e)
2. Ionic Polarization (P_e)
3. Orientation Polarization (P_o)
4. Space charge Polarization (P_s)

(1) Electronic Polarization:

E = 0 **E > 0**

This polarization involves the separation of the center of the electron cloud around an atom with respect to the center of its nucleus in dielectric material under the application of electric field. Hence dipoles are induced within the material. This leads to the development of net dipole moment in the material and is the vector sum of dipole moments of individual dipoles.

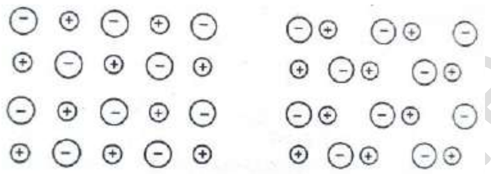
It is independent of the temperate.

The expression for electronic polarizability is given by

$$\alpha_e = \frac{\varepsilon_0(\varepsilon_r - 1)}{\mathrm{N}} = 4\pi\varepsilon_0 R^3$$

where N is the number of free electrons per unit volume.

(2) Ionic Polarization:



E = 0 **E > 0**

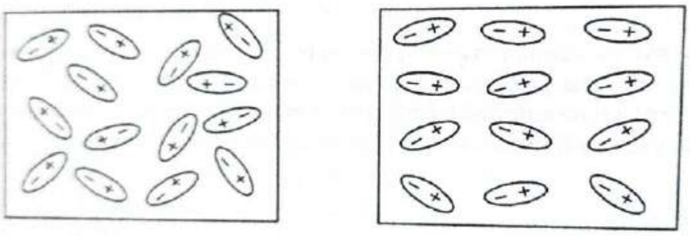
This polarization occurs in ionic solids such as sodium chloride etc., Ionic solids possess net dipole moment even in the absence of external electric field. But when the external electric field is applied the separation between the ions further increases. Hence the net dipole moment of the material also increases.

The Ionic polarization occurs in ionic molecules such as HCl, NaCl, etc., It is also temperature independent quantity.

The expression for ionic polarizability is given by

$$\alpha_i = \frac{P_e}{nE} = \frac{1}{10}\alpha_e$$

(3) Orientation Polarization:



E = 0 **E > 0**

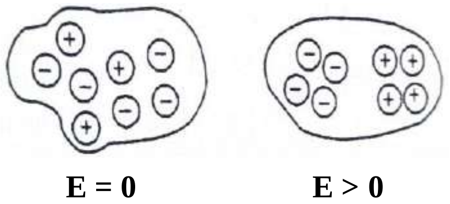
This polarization occurs in polar dielectric material, which possesses permanent electric dipoles. In polar dielectrics the dipoles are randomly oriented due thermal agitation. Therefore, net dipole moment of the material is zero. But when the external electric field is applied all dipoles tend to align in the field direction. Therefore, dipole moment develops across the material. This is referred to as orientation polarization.

It is dependent on temperature. i.e., it is inversely proportional to the temperature.

The expression for orientation polarizability is given by

$$\alpha_o = \frac{\mu^2}{3kT}$$

(4) Space Charge Polarization:



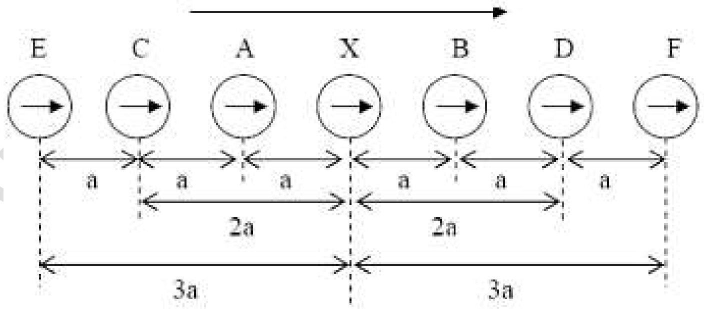
This polarization involves limited movement of charges resulting in alignment of charge dipoles under applied field. This occurs in multiphase dielectric materials in which there is a change of resistivity between different phases. This usually happens at the grain boundaries or lattice defects and localized charge is set up. Since this is very small it can be neglected.

Internal fields in Dielectrics

Internal field - Definition:

When a dielectric material is placed in the external electric field, polarization occurs. Hence the net electric field at any point within the dielectric material is given by the sum of external field and the field due to all dipoles surrounding that point. This net field is called internal field or Local field.

Expression for Internal field in case of one dimension



Consider a dielectric material placed in the external electric field E. Consider an array of dipoles in the polarized dielectric material as shown in the figure. Let a be the distance between two successive dipoles in the array.

The internal field at a dipole 'X' due to all dipoles in the array is given by

$$E_i = E + \frac{1.2\mu}{\pi\varepsilon_0 a^3} \quad \rightarrow (1)$$

where μ is the dipole moment, ε_0 is the permittivity of free space and a is the inter dipole distance in the array.

$$E_i = E + \frac{1.2\alpha E}{\pi\varepsilon_0 a^3} \quad \rightarrow (2)$$

here α is polarizability. Since α, ϵ_0 and a are positive quantities the local field $Ei > E$.

Expression for Internal field in case of three dimensional array of atoms - Lorentz Field

For three dimension $\frac{1}{a^3}$ could be replaced with number of dipoles per unit volume N and $\frac{1.2}{\pi}$ could be replaced with γ in equation (2), we get

$$E_i = E + \frac{\gamma N \alpha E}{\varepsilon_0} \quad \rightarrow (3)$$

$$E_i = E + \frac{\gamma P}{\varepsilon_0} \quad \rightarrow (4)$$

Here the polarization $P = N\alpha E$.

For an elemental solid dielectric material $\gamma = 1/3$, Thus equation (4) becomes

$$E_L = E + \frac{P}{3\varepsilon_0} \quad \rightarrow (5)$$

Here $E_i = E_L$ and hence the above equation is the expression for Lorentz field.

Derivation of Clausius - Mossotti equation:

Consider an Elemental solid dielectric material. Since it doesn't possess permanent dipoles, for such material, the ionic and orientation polarizabilities are zero. Hence the polarization P is given by

$$P = N\alpha_e E_L \quad \rightarrow (1)$$

When we substitute the value of $E_L = E + \frac{P}{3\varepsilon_0}$ in the above equation (1) we get,

$$P = N\alpha_e \left(E + \frac{P}{3\varepsilon_0} \right) \quad \rightarrow (2)$$

Again substitute the value of $P = \varepsilon_0(\varepsilon_r - 1)E$ in the above equation, then

$$\varepsilon_0(\varepsilon_r - 1)E = N\alpha_e \left(E + \frac{\varepsilon_0(\varepsilon_r - 1)E}{3\varepsilon_0} \right)$$

$$\varepsilon_0(\varepsilon_r - 1)E = N\alpha_e \left(E + \frac{(\varepsilon_r - 1)E}{3} \right)$$

$$\varepsilon_0(\varepsilon_r - 1)E = N\alpha_e E \left(1 + \frac{(\varepsilon_r - 1)}{3} \right)$$

$$\varepsilon_0(\varepsilon_r - 1) = N\alpha_e \left(\frac{3 + (\varepsilon_r - 1)}{3} \right)$$

$$(\varepsilon_r - 1) = N\alpha_e \frac{(\varepsilon_r + 2)}{3\varepsilon_0}$$

$$\frac{N\alpha_e}{3\varepsilon_0} = \frac{(\varepsilon_r - 1)}{(\varepsilon_r + 2)} \quad \rightarrow (3)$$

Equation (3) is called Clausius-Mossotti relation or equation.

Solid, liquid and gaseous dielectrics:

Two most important applications of dielectric materials are as insulation material and as medium in capacitors. For insulation materials application the dielectric is required to have low dielectric constant, low dielectric loss, high resistance and high dielectric strength.

Solid insulating materials: - Polymers and ceramics are the widely used Solid insulators. A variety of plastics, rubbers, waxes, paper, synthetic fibers and fabrics are applied in the form of films, sheets, slabs, tapes, sleeving, tubing, rods and mouldings.

Liquid insulating materials: - Liquid insulating materials are mainly mineral oils and synthetic oils, which are used for the purpose of insulation as well as cooling in transformers.

Gaseous insulating materials: - Gases are good insulators and work well as heat transferring media. E.g., air, nitrogen, inert gases, hydrogen, CO_2, etc.

Application of dielectrics in transformers: - A transformer is a device used for transmitting power from one circuit to another or from one place to another place. It consists of two windings, primary and secondary windings, linked by a common magnetic flux. During the construction of transformers, the windings are filled (or saturated) by varnishes. In case of H.V. transformers used in distribution of power where very high voltages are present, proper provisions are to be provided to distribute away the heat produced and to provide high dielectric strength. These transformers are usually immersed in liquid dielectrics.

Mineral oil transfers heat from the transformers windings and core to the outer shield and enables dissipation of the heat generated. Nowadays, synthetic oils are being used in place of mineral oils because synthetic oils are much more resistant to oxidation and fire hazards.

Petroleum oils, silicone oils and vegetable oils are having high thermal stability. They are mainly used as filling medium for transformers, circuit breakers etc.

Problems:

1) Find the polarization produced in a dielectric medium of relative permittivity 15 in presence of an electric field of 500 V/m.

Soln.: p=?, $\varepsilon_r = 15$, E=500 v/m

$$P = \varepsilon_0(\varepsilon_r - 1)E$$

$$P = 8.854\times10^{-12}(15-1)500$$

$$P = 6.198\times10^{-8}\,Cm^{-2}$$

2) The dielectric constant of He gas at NTP is 1.0000684. Calculate the electronic polarizabilty of He atoms if the gas contains $2.7\text{x}10^{25}$ atoms/m^3.

Soln. : $\varepsilon_r = 1.0000684$, α_e =?, N = $2.7\text{x}10^{25}$ atoms/m^3.

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N\alpha_e}{3\varepsilon_0} \qquad \text{or} \qquad \alpha_e = \frac{3\varepsilon_0(\varepsilon_r - 1)}{N(\varepsilon_r + 2)}$$

$$\alpha_e = \frac{3 \times 8.854 \times 10^{-12} \times (1.0000684 - 1)}{2.7 \times 10^{25} \times (1.0000684 + 2)}$$

$$\alpha_e = 2.243 \times 10^{-41} Fm^2$$

(3) The dielectric constant of sulphur is 3.4 assuming a cubic lattice for its structure, calculate the dielectric constant of sulphur. Given density = 2.07 gm/cc and atomic weight = 32.07.

Soln.: $\varepsilon_r = 3.4$, α_e = ?, D = 2.07 gm/cc = 2.07x10^3 kg/m^3, A = 32.07

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N\alpha_e}{3\varepsilon_0}$$

and

$$N = \frac{no.of\ \ free\,electron \times N_A \times D}{A}$$

$$\therefore \frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N_A \times D \times \alpha_e}{3\varepsilon_0 A}$$

$$\therefore \alpha_e = \frac{3\varepsilon_0 A(\varepsilon_r - 1)}{N_A \times D(\varepsilon_r + 2)}$$

$$\therefore \alpha_e = \frac{3 \times 8.854 \times 10^{-12} \times 32.07 \times (3.4 - 1)}{6.025 \times 10^{26} \times 2.07 \times 10^3 \times (3.4 + 2)}$$

$$\alpha_e = 3.035 \times 10^{-40} Fm^2$$

4) An elemental solid dielectric material has polarizability $\alpha_e = 7 \times 10^{-40} Fm^2$ **. Assuming the internal field to be Lorentz field, calculate the dielectric for the material if the material has** $N = 3 \times 10^{28} atoms/m^3$.

Soln.: $\alpha_e = 7 \times 10^{-40} Fm^2, \varepsilon_r = ?, N = 3 \times 10^{28} atoms/m^3$

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N\alpha_e}{3\varepsilon_0}$$

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{3 \times 10^{28} \times 7 \times 10^{-40}}{3 \times 8.854 \times 10^{-12}}$$

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = 0.7906$$

$$\varepsilon_r - 1 = 0.7906\varepsilon_r + 1.5812$$

$$\varepsilon_r(1 - 0.7906) = 1 + 1.5812$$

$$\varepsilon_r = \frac{2.5812}{0.2094} = 12.33$$

Numerical Problems:

1. Determine the polarization produced in a crystal by an electric field of strength 6000 V/cm if it has a dielectric constant of 5. Ans. $2.124\times10^{-5}\,Cm^{-2}$

2. Find the polarization produced in a dielectric medium of relative permittivity 15 in presence of an electric field of 500 V/m.

2\. Calculate the dielectric constant of NaCl crystal if the polarization developed is 4.3×10^{-8} when subjected to electric field of 2000 Vm^{-1}. Ans. 3.428.

3\. An elemental solid dielectric material has polarizability $7\times10^{-40}\,Fm^{-2}$. Assuming the internal field to be Lorentz, calculate the dielectric constant for the material if the material has $3\times10^{28}\,atoms\,/m3$. Ans. 12.33.

4\. The dielectric constant of sulphur is 3.4. Assuming a cubic lattice for its structure, calculate the electronic polarizability of sulphur. Given: for sulphur density = 2.07 gm/cc, and atomic weight 32.07. Ans. $\alpha_e = 3.035\times10^{-40}\,Fm^2$.

5\. The atomic weight and density of sulphur are 32 and $2.08\times10^3\,kg/m^3$. The electronic polarizability of the atom is $3.28\times10^{-40}F/m^2$. If Sulphur solid has a cubic structure, calculate its dielectric constant. Ans. 3.873.

6\. A solid contains $5\times10^{28}\,atoms\,/\,m^3$ each with a polarizability of $2\times10^{-40}\,Fm^{-2}$. Assuming that the internal field is given by Lorentz formula. Calculate the ratio of internal field to the external field. Given $\epsilon_0 = 8.854\times10^{-12}\,Fm^{-1}$.

Review Questions:

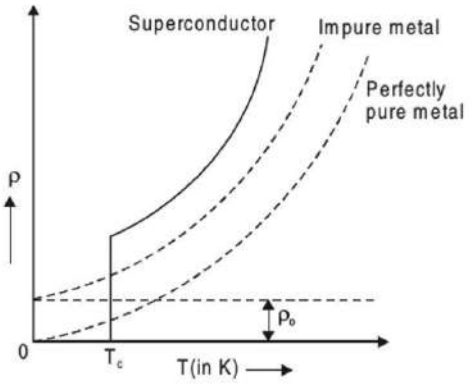
1. Define the terms electric dipole moment, dielectric constant and Electric polarization.
2. Define dipole, dipole moment and define dielectric constant.
3. What are Polar and Non polar dielectrics? Explain with examples.
4. Derive relation between polarization, electric susceptibility and dielectric constant.
5. Define polarization of a dielectric material and mention the types of polarizations.
6. Describe in brief the various types of polarization mechanisms.
7. What is internal field? Write an expression for internal field in case of one-dimensional and three dimensional array of atoms in dielectric solids.
8. Derive Clausius-Mossoti equation.
9. What are dielectrics? Explain electronic and ionic polarization. Give an example for each.
10. What are dielectrics? Explain ionic and orientation polarization. Give an example for each.

Superconductivity:

Introduction: Lord Kamerlingh Onnes discovered the phenomenon of superconductivity in the year 1911. When he was studying the temperature dependence of resistance of Mercury at very low temperature he found that resistance of Mercury decreases in temperature up to a particular temperature T_c = 4.15 K and below this temperature the resistance of mercury abruptly drops to zero. Between 4.15K and 0K Mercury offered no resistance for the flow of electric current. This phenomenon is reversible and material becomes normal when once again temperature was increased above 4.15 K. This phenomenon is called superconductivity and material which exhibits this property is named as superconductor.

Definition: Superconductivity is defined as "The phenomenon in which resistance of certain metals, alloys and compounds drops to zero abruptly, below certain temperature is called superconductivity.

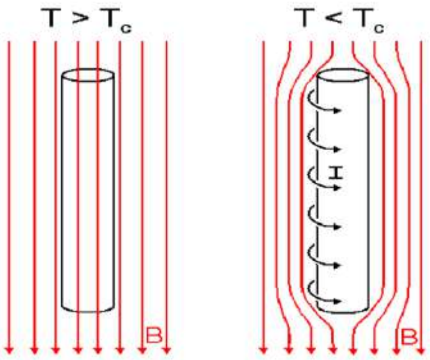
Variation of Resistivity with Temperature: The variation of the resistivity of a superconductor, pure and impure metals with temperature is as shown in the figure below.



Critical Temperature: The temperature, below which materials exhibit superconducting property is called critical temperature, denoted by T_C. It is different for different substances. The materials, which exhibit superconducting property, are called superconductors. Above critical temperature material is said to be in normal state and offers resistance for the flow of electric current. Below critical temperature material is said to be in superconducting state. Thus T_C is also called as transition temperature.

Meissner's Effect

In 1933, Meissner and Ochsenfeld showed that when a superconducting material is placed in a magnetic field it allows magnetic lines of force to pass through, if its temperature is above T_C. If the temperature is reduced below the critical temperature T_C, then it expels all the flux lines completely out of the specimen and exhibits perfect diamagnetism. This is known as Meissner's effect. Since superconductor exhibits perfect diamagnetism below the critical temperature Tc, magnetic flux density inside the material is zero.



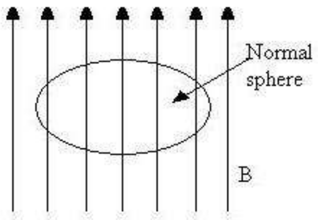


Fig (b): T > Tc

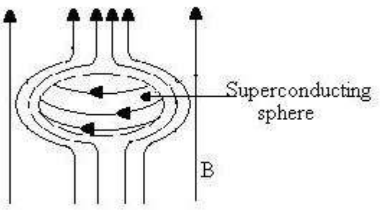


Fig (c): T < Tc

The expression for magnetic flux density is given by $B = \mu_0 (M + H)$

Here B is Magnetic Flux Density, M is Magnetization and H is the applied magnetic field strength. For a superconductor, $B = 0$ at $T < T_C$.

Thus we get $M = -H$.

Thus Meissner's Effect signifies the negative magnetic moment associated with superconductors.

Critical Field and its Temperature Dependence

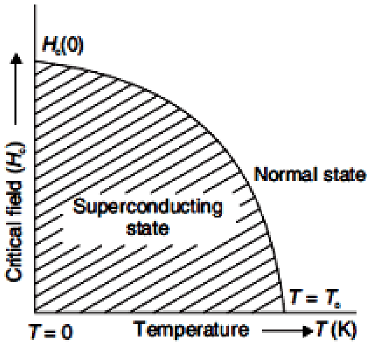
Critical field We know that when superconductor is placed in a magnetic field it expels magnetic flux lines completely out of the body and exhibits a perfect diamagnetism. But if the strength of the magnetic field is further increased, it is found that for a particular value of the magnetic field,

material looses its superconducting property and becomes a normal conductor. The value of the magnetic field at which the transition occurs from the Superconducting state to Normal Conducting state is called Critical Field or Critical Magnetic Field and is denoted by H_C. It is found that by reducing the temperature of the material further superconducting property of the material could be restored. Thus, critical field does not destroy the superconducting property of the material completely but only reduces the critical temperature of the material.

The variation of Critical field with temperature below the critical temperature is given by

$$H_C = H_0\left(1 - \frac{T^2}{T_C^2}\right)$$

Here H_C is the Critical field at any temperature T less than T_C, H_0 is the Critical field at T = 0 K.



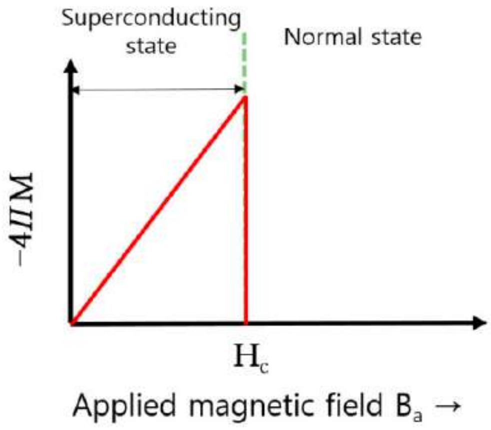
Types of Superconductors

Superconductors are classified into two types

1. Type - I Superconductor or Soft Superconductor
2. Type - II Superconductor or Hard Superconductor

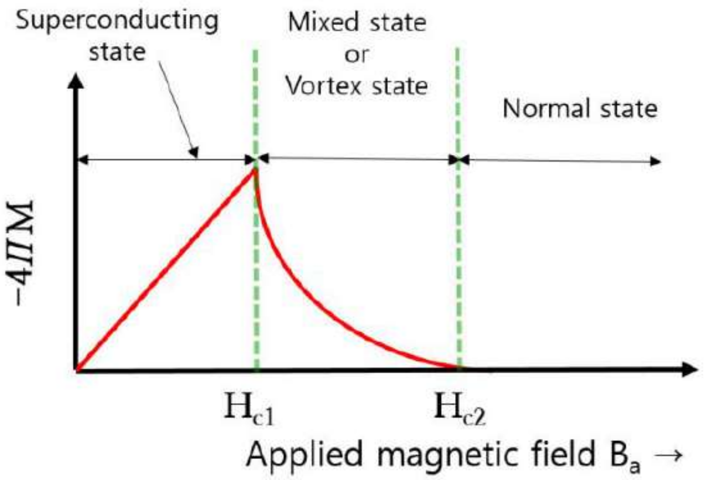
Type - I Superconductors:

Type I superconductors exhibit complete Meissener's Effect and possess a single value of critical field. The graph of magnetic moment Vs magnetic field is as shown in the Fig. below. As the field strength increases the material becomes more and more diamagnetic until H becomes equal to H_C. Above H_C the material allows the flux lines to pass through and exhibits normal conductivity. The value of H_C is very small for soft superconductors. Therefore, soft superconductors cannot withstand high magnetic fields. Therefore, they cannot be used for making superconducting magnets. Ex. Hg, Pb and Zn.

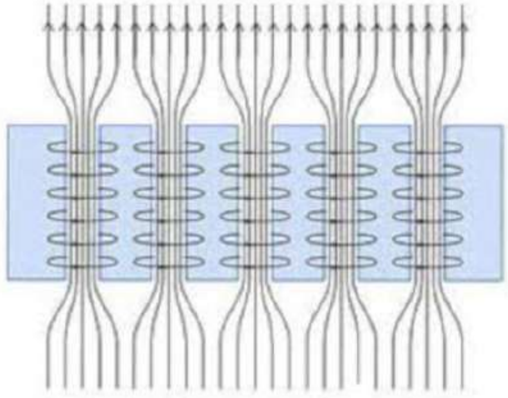


Type - II Superconductors:

paragraph Superconducting materials, which can withstand high value of critical magnetic fields, are called Hard Superconductors.



The graph of magnetic moment Vs magnetic field is as shown in the Fig. Hard superconductors are characterized by two critical fields H_{C1} and H_{C2}. When applied magnetic field is less than H_{C1} material exhibits perfect diamagnetism. Beyond H_{C1} partial flux penetrates and the material is said to be Vortex State. Thus flux penetration occurs through small-channelized regions called filaments. As the strength of the field increases further, more and more flux fills the body and thereby decreasing the diamagnetic property of the material. At H_{C2} flux fills the body completely and material losses its diamagnetic property as well as superconducting property completely.



The value of H_{C2} is hundreds of times greater than H_C of soft superconductors. Therefore, they are used for making powerful superconducting magnets. Ex. : $NbTi$, Nb_3Sn.

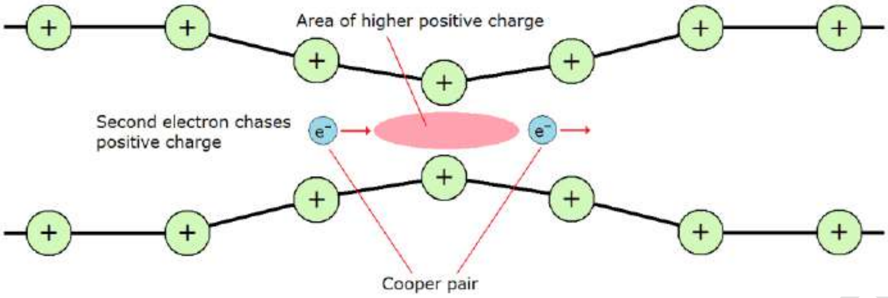
Difference between Type - 1 and Type -2 Superconductors	
Type - 1 Superconductor	**Type -2 Superconductor**
1. They exhibit complete Meissner effect	1. They exhibit partial Meissner effect
2. These are perfect diamagnetics	2. These are not perfect diamagnetics
3. These are known as soft superconductors	3. These are known as hard superconductors
4. They have only one critical magnetic field	4. They have two critical magnetic fields
5. No vertex (Mixed) state is present	5. Vertex (mixed) state is present
6. These materials undergoes a sharp transition at the critical magnetic field	6. These materials undergoes a gradual transition between two critical magnetic fields
7. The highest value of critical magnetic field is 0.1 wb/m^2	7. The upper critical magnetic field is of the 50 wb/m^2
8. Critical temperature is low (< 10 K)	8. Critical temperature is high (> 10 K)
9. Applications are very limited	9. They are used to generate very high magnetic field.
10. Examples:- lead, tin, mercury , etc.	10. Examples:- alloys like Nb-Sn, Nb-Ti, Nb-Zr, etc.

BCS Theory of Superconductivity:

Bardeen, Cooper and Schrieffer explained the phenomenon of superconductivity in the year 1957. The essence of the BCS theory is as follows.

Consider an electron approaching a positive ion core and suffers attractive coulomb interaction. Due to this attraction ion core is set in motion and thus distorts that lattice. Let a second electron come in the way of distorted lattice and interaction between the two occurs which lowers the energy of the second electron. The two electrons therefore interact indirectly through the lattice distortion or the phonon field which lowers the energy of the electrons. The above interaction is interpreted as electron - Lattice – electron interaction through phonon field.

It was shown by Cooper that, this attractive force becomes maximum if two electrons have opposite spins and momentum. The attractive force may exceed coulombs repulsive force between the two electrons below the critical temperature, which results in the formation of bound pair of electrons called cooper pairs.



Below the critical temperature the dense cloud of Cooper pairs form a collective state and the motion all Cooper pairs is correlated resulting in zero resistance of the material.

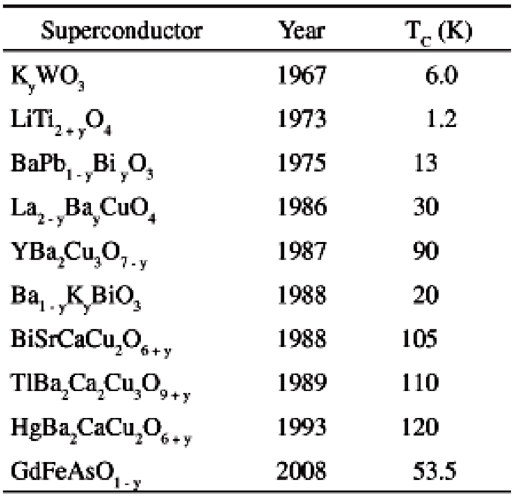
High Temperature Superconductors:

Superconducting materials which exhibit superconductivity at relatively higher temperatures are called high temperature superconductors. Thus high temperature superconductors possess higher value of critical temperature compared to conventional superconductors. Most of the high temperature superconductors are found to fall into the category of ceramics. In 1986 George Bednorz and Alex Muller discovered a compound containing Lanthanum, Barium, Copper and Oxygen having $T_C = 30$ K was developed. In 1987 scientists developed a compound which is an oxide of the form $YBa_2Cu_3O_7$ which is referred to as 1-2-3 compound with $T_C > 90$ K was discovered.

All high temperature superconductors are oxides of copper and bear Perovskite crystal structure characterized by large number of copper-oxygen layers. It was found that addition of extra copper-oxygen layer pushes the critical temperature T_C to higher values. The super currents are strong in the copper-oxygen layer and weak in the direction perpendicular to the planes.

High Temperature superconductors are not commercially available mainly due to their current densities and difficulty in forming into wires. Once these difficulties are overcome such superconductors find many applications in various fields like zero-loss power transmission lines, super- strong magnetic materials and as the materials for levitating trains.

Following is the list of some of High Temperature Superconductors.

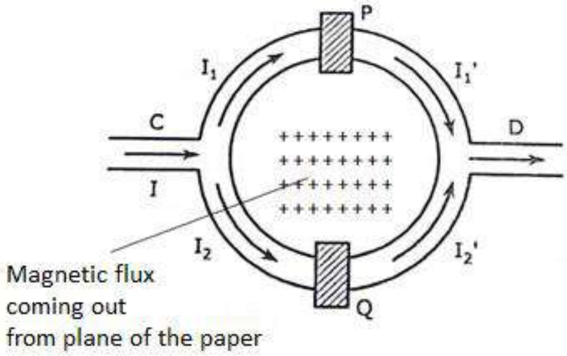


Superconductor	Year	T_C (K)
K_yWO_3	1967	6.0
$LiTi_{2+y}O_4$	1973	1.2
$BaPb_{1-y}Bi_yO_3$	1975	13
$La_{2-y}Ba_yCuO_4$	1986	30
$YBa_2Cu_3O_{7-y}$	1987	90
$Ba_{1-y}K_yBiO_3$	1988	20
$BiSrCaCu_2O_{6+y}$	1988	105
$TlBa_2Ca_2Cu_3O_{9+y}$	1989	110
$HgBa_2CaCu_2O_{6+y}$	1993	120
$GdFeAsO_{1-y}$	2008	53.5

DC - SQUID:

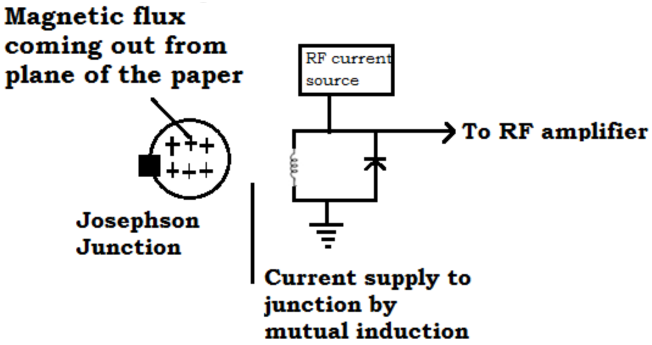
It has two Josephson junctions connected in parallel and works on the interference of current from two junctions. It works on the principle of DC Josephson effect which is the phenomenon of flow of super current through the junction even in the absence of external emf or voltage.

Construction and working:



The cross sectional view of the arrangement is shown. P and Q are two Josephson Junctions arranged in parallel. When current I flows through the point C, it divides into I_1 and I_2. Hence the wave function due to these super currents (cooper pairs) experiences a phase shift at P and Q. In the absence of applied magnetic field, the phase difference between the wave functions is zero. If the magnetic field is applied perpendicular to the current loop, then phase difference between the wave functions will not be zero. This is identified by the sum of the currents I_1 and I_2. The magnitude of phase difference is proportional to applied magnetic field. Hence, even if there is a weak magnetic field in the region will be detected.

RF SQUID:



It works on the principle of AC Josephson effect. When dc voltage is applied across the Josephson junction, it leads to the development of oscillating current. It has single Josephson junction. Magnetic field is applied perpendicular to the plane of the current loop. The flux is coupled into a loop containing a single Josephson junction through an input coil and an RF source. Hence when the RF current changes, there is corresponding change in the flux linked with the coil. This variation is very sensitive and is measured. It is also used in the detection of low magnetic field. It is less sensitive compared to DC - SQUID. Due to its low cost manufacturing, it is commonly used in many applications.

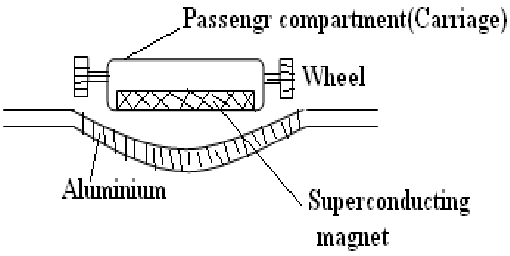
Problem: - A superconducting tin has a critical field of 306 gauss at 0K and 217 gauss at 2K. Find the critical temperature of superconducting tin.

Data: - H_0=306gauss at 0K, H_C=217gaussat 2K=T

$$H_C = H_0\left(1 - \frac{T^2}{T_c^2}\right)$$

$$T_C = \frac{T}{\sqrt{1 - \frac{H_c}{H_0}}} = \frac{2}{\sqrt{1 - \frac{217}{306}}} = 3.71\,K$$

Maglev vehicles:



Magnetically Levitated vehicles are called maglev vehicles. Since such vehicles float under magnetic effect they offer benefit like no friction, less power and noiseless transportation. The phenomenon on which the magnetic levitation is based is Meissner Effect.

It uses direct current superconducting magnets mounted under the carriage in turn induce eddy currents in an aluminium, thus generating repulsive forces, which in turn lift the carriage. (A computer changes the amount of current to keep the train 1 cm from the track. This means there is no friction between the train and the track) Then the forward motion of the vehicle is achieved by the principle of synchronous linear induction (A maglev's guide way has a long line of electromagnets. These pull the train from the front and push it from behind). The vehicle does not levitate until it reaches 50 mph, so it is equipped with retractable wheels. The wheels serve almost the same purpose as those of an aero plane.

A prototype maglev train has been constructed in Japan, which reached speeds of about 430Km/hr. German made maglev train is operating between airport and Pudong district in the Chinese city of Shanghai. The superconducting magnets were cooled with liquid helium. Linear generators will produce all the electricity needed in the train's interior. Only the part of the track that is used will be electrified so no energy is wasted.

Q. No.	Questions
1.	State and explain Meissner Effect.
2.	Define critical field and hence explain its variation with temperature below critical temperature.
3.	Explain Type -1 Type – 2 superconductors with neat figures.
4.	Distinguish between Type -1 Type – 2 superconductors.
5.	Describe high temperature superconductors.
6.	Elucidate the BCS theory of superconductivity.
7.	Explain the phenomenon of quantum tunneling.
8.	Define Josephson junction and hence explain DC and AC Josephson effects.
9.	Define SQUID and describe DC and RF SQUIDs.
10.	Brief out the applications of superconductivity in quantum computing.

Q. No.	Numerical problems
1.	Lead has superconducting transition temperature of 7.26 K. If the initial field at 0 k is 50x103 A/m, calculate the critical field at 6 K.
2.	A superconducting tin has a critical temperature of 3.7 K at zero magnetic field and a critical field of 0.0306 tesla at 0 K. Find the critical field at 2 K.
3.	The superconducting transition temperature of lead is 7.26 K. Calculate the initial field at 0 K, given the critical field at 5 K is 33.644x103 A/m.
4.	Calculate the ratio of critical fields for a superconductor at 7 K and 5K, given the critical temperature is 8 K.
5.	The critical field for niobium is 1x105 A/m at 8 K and 2x105 A/m at 0 K. Calculate the transition temperature of the element.

* * * * ***END*** * * * *

MYSURU ROYAL INSTITUTE OF TECHNOLOGY

DEPARTMENT OF PHYSICS

Applied Physics for EEE Stream : 2022-23

Module – 3 : Laser & Optical Fibers

Syllabus:

Laser and Optical Fibers: 8 Hrs

Lasers: Characteristics of LASER, Interaction of radiation with matter, Expression for Energy Density and its significance. Requisites of a Laser System. Conditions for Laser action. Principle, Construction and Working of Carbon Dioxide Laser. Application of Lasers in Defense (Laser range finder) and Laser Printing. Numerical Problems

Optical Fibers: Total Internal Reflection**,** Propagation mechanism, Angle of Acceptance, Numerical Aperture, Fractional Index Change, Modes of Propagation, Number of Modes and V Number, Types of Optical Fibers. Attenuation and Mention of Expression for Attenuation coefficient, Attenuation Spectrum of an Optical Fiber with Optical Windows. Discussion of Block Diagram of Point to Point Communication, Intensity based Fiber Optic Displacement Sensor, Merits and Demerits, Numerical problems.

Pre-requisite: Properties of light

Self-learning: Total Internal Reflection

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LASER

Introduction:

LASER is the acronym (short form) for **L**ight **A**mplification by **S**timulated **E**mission of **R**adiation. The important Characteristic properties of LASER beam are

1. It is highly monochromatic.
2. It is highly coherent.
3. It is highly directional.
4. It is a high intensity beam of light.

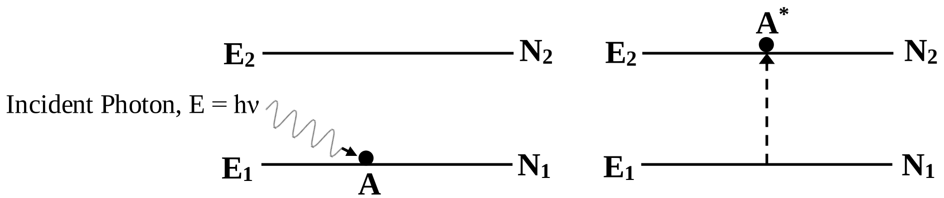
In order to understand the mechanism involved in the production of laser beam, one has to know the process taking place in an atomic system such as absorption and Emission of radiation.

Interaction of radiation with matter:

The interaction of radiation with matter occurs through the following three processes, namely.

1. Induced absorption
2. Spontaneous emission and
3. Stimulated emission

1. Induced Absorption:



(*Before*) $A + h\nu \rightarrow A^*$ (After)

When a suitable energy of a photon is incident on an atom, the photon is absorbed it. In this process the incident photon excites an atom from ground state to higher energy sate and hence it is known as Induced Absorption.

Consider an atom in a lower energy states E_1, it will excite to higher energy states E_2 by absorbing the incident photon of energy $E = h\nu = E_2 - E_1$. where h is the Planck's constant and ν is the frequency of the incident photon. The induced absorption can be represented as

$$A + h\nu \rightarrow A^*$$

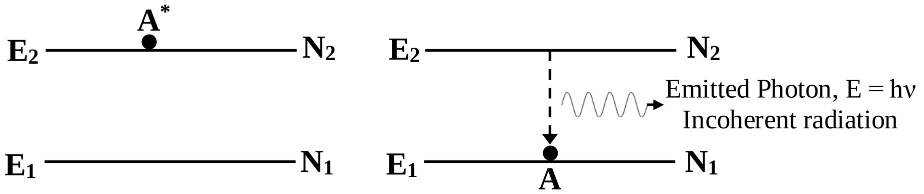
Where A is the atom in the ground sate E_1 and A* is the excited atom in the higher energy sate E_2.

Let N_1 and N_2 be the number of atoms in the energy levels E_1 and E_2, and U_ν be the energy density of the incident radiation. Thus the probability of transition of atoms from E_1 to E_2 is depending on N_1 and U_ν .

Therefore, the rate of induced absorption is $\propto N_1U_\nu = B_{12}N_1U_\nu$

Where B_{12} is a proportionality constant known as the Einstein's coefficient for induced absorption.

2. Spontaneous emission:



$(Before)\quad A^* \rightarrow A + h\nu\quad$ (After)

Spontaneous emission is the process of emission of photon, when an atom transits from higher energy level to lower energy level without the influence of any external energy.

An atom in the higher energy state E_2 makes a transition to lower energy state E_1 without the action of any external agency. the photon of energy $E_2 - E_1 = h\nu$ is emitted. In this process the emitted photons need not travel in the same direction. Thus the emitted light beam is not directional. The spontaneous emission can be represented as

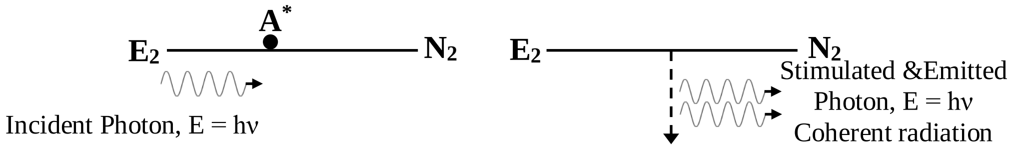
$$A^* \rightarrow A + h\nu$$

In this process, the probability of transition of atoms from E_2 to E_1 is depending on N_2 only.

Therefore, the rate of spontaneous emission is $\propto N_2 = A_{21}N_2$

Where A_{21} is a proportionality constant known as the Einstein's coefficient for spontaneous emission.

3. Stimulated emission:



$(Before)\quad A^* + h\nu \rightarrow A + 2h\nu\quad$ (After)

When a photon of suitable energy interacts with an atom in the higher energy state without loose energy then the atom is stimulated (Forced) to make transition from higher energy state E_2 to the lower energy state E_1 with the emission of a photon of energy $E_2 - E_1 = h\nu$. Both the incident photon and the emitted photons are coherent or in phase and travel in the same direction. This process is known as stimulated emission. The stimulated or induced emission can be represented as

$$A^* + h\nu \rightarrow A + 2h\nu$$

This kind of emission is responsible for laser action.

In this process, the probability of transition of atoms from E_2 to E_1 is depending on N_2 and U_ν

Therefore, the rate of stimulated emission is $\propto N_2U_\nu = B_{21}N_2U_\nu$

Where B_{21} is a proportionality constant known as the Einstein's coefficient for stimulated emission.

Expression for Energy density in terms of Einstein Coefficients:

Consider a system containing of large number of atoms is under thermal equilibrium. Let E_1 and E_2 be the lower and higher energy levels which contains N_1 and N_2 number of atoms respectively. Let U_ν be the energy density of the incident radiation. Hence the system absorbs and emits the energy through the processes of induced absorption spontaneous emission and stimulated emission. The energy of the photon absorbed and emitted by the atoms is $E = h\nu = E_2 - E_1$.

The system be in thermal equilibrium; the total energy of the system remains unchanged in spite of the interaction of the incident radiation.

At thermal equilibrium,

Rate of Absorption = Rate of Spontaneous emission + Rate of Stimulated emission

∴ we have,

$$B_{12}N_1U_\nu = A_{21}N_2 + B_{21}N_2U_\nu$$

$$B_{12}N_1U_\nu - B_{21}N_2U_\nu = A_{21}N_2$$

$$U_\nu\left(B_{12}N_1 - B_{21}N_2\right) = A_{21}N_2$$

$$U_\nu = \frac{A_{21}N_2}{\left(B_{12}N_1 - B_{21}N_2\right)}$$

÷ both numerator & denominator of the above equation by B_{21} and N_2, we get,

$$U_\nu = \frac{A_{21}}{B_{21}}\left(\frac{1}{\frac{B_{12}N_1}{B_{21}N_2} - 1}\right) \qquad \rightarrow (4)$$

According to Boltzmann distribution law, we have

$$N_i = N_0 e^{\frac{-E_i}{KT}} \qquad \therefore \frac{N_1}{N_2} = \frac{e^{-\frac{E_1}{KT}}}{e^{-\frac{E_2}{KT}}} = e^{\frac{E_2-E_1}{KT}} = e^{\frac{h\gamma}{KT}} \text{ OR } \frac{N_2}{N_1} == e^{-\frac{h\nu}{KT}} \qquad \rightarrow (5)$$

Now, by substituting this in equation (4), we have

$$U_\nu = \frac{A_{21}}{B_{21}}\left(\frac{1}{\frac{B_{12}}{B_{21}}e^{\frac{h\nu}{KT}} - 1}\right) \qquad \rightarrow (6)$$

According to Planck's law, the equation for energy density E_γ is given by

$$U_\nu = \frac{8\pi h\nu^3}{c^3}\left(\frac{1}{e^{\frac{h\nu}{KT}} - 1}\right) \qquad \rightarrow (7)$$

Now, comparing equation (6) and equation (7), we have

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3} \text{ \& } \frac{B_{12}}{B_{21}} = 1 \text{ or } B_{12} = B_{21}$$

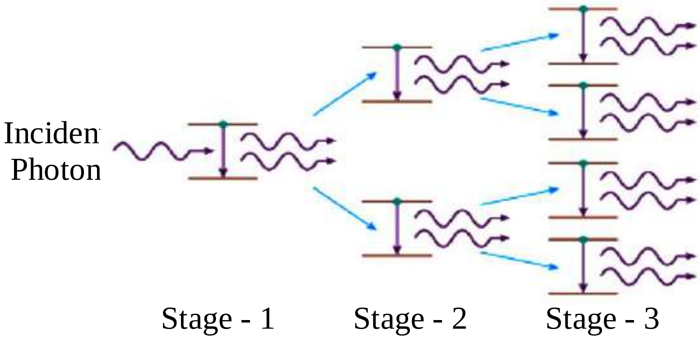
This implies that the probability of induced absorption is equal to the probability of stimulated emission. Because of the above identity, A_{21} and B_{21} can be represented by A and B in equation (6) and it can be rewrite as.

$$U_\nu = \frac{A}{B}\left(\frac{1}{e^{\frac{h\nu}{KT}}-1}\right)$$

This is the expression for energy density in terms of Einstein's A and B coefficients.

Condition for Lasing action:

Consider a LASER system. Let an atom in the excited state is stimulated by a photon of suitable energy, so that atom makes stimulated emission. Then two coherent photons are obtained. These two coherent photons if stimulate two more atoms in the exited state to make emission then four coherent photons are obtained. These four coherent photons stimulate four more atoms in the excited state resulting in eight coherent photons and so on. As this process continues, number of coherent photons produced increases. These coherent photons constitute an intense beam of LASER. This phenomenon of building up of number of coherent photons, so as to get an intense LASER beam is called lasing action.



LASER action could be achieved through the conditions of population inversion by pumping and meta-stable state.

(i) Population Inversion:

When a system is under thermal equilibrium, the number of atoms in excited state is less than the number of atoms in the lower energy state. The production of LASER is achieved through stimulated emission rather than induced absorption and spontaneous emission. This is possible only if the number of atoms in the higher energy state is more than the number of atoms in the lower energy state and the process of achieving this is called population inversion.

Thus the essential conditions for population inversion are

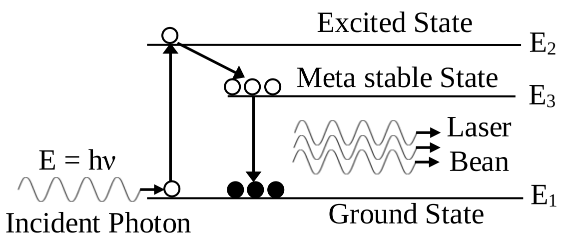
a) Higher energy state should possess a longer life time.

b) The number of atoms in the higher energy state must be greater than the number of atoms in the lower energy state.

(ii) Pumping Process:

Population inversion is achieved by supplying energy from a suitable source is called Pumping. In addition, to have more stimulated emissions, the life time of atoms in the excited state must be longer. There are number of techniques for pumping a collection of atom to an inverted state (excited state).

(iii) Meta stable state:



The life time of an energy level is of the order of 10^{-8} second. If an atom possesses unusual longer life time in an energy state such a state is referred to as a meta-stable state. Usually the life time of meta-stable state varies from 10^{-2}s to 10^{-3}s. Population inversion could be achieved with the help of three energy state with one of them a meta-stable state and is as shown in the figure. The population inversion is achieved between the state E_2 and E_1 as state E_3 is a meta-stable state.

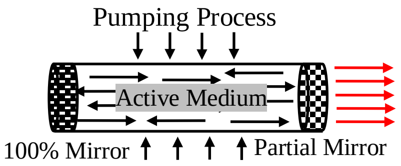
Note : The principles of Laser are 1. Stimulated Emission, 2. Population Inversion & 3. Meta-stable State

Requisites of a LASER system:

The three requisites of a LASER system are;

1. Active medium
2. Pumping Source
3. LASER cavity

1. Active medium: Population inversion occurs at certain stage in the active medium due to the absorption of energy. The active medium supports meta-stable states. After this stage the active medium is capable of emitting LASER light.



2. Pumping Source: Pumping source is an excitation source in order to achieve population inversion. That means more and more atoms are to be moved to higher energy state is called pumping. This is achieved by supplying suitable energy from an energy source. optical / light energy is used in optical pumping and electrical energy is used electrical pumping.

3. LASER Cavity: The LASER Cavity is an active medium bound between two highly parallel mirrors. The reflection of photons from the mirrors results in multiple traverse of photons through the active medium inducing more and more stimulated emissions. Thus amplification of light is achieved. This also helps to tap certain permissible part of LASER energy from the active medium. The cavity resonates and the output will be maximum when the distance L between the mirrors is equal to an integral multiple of $\lambda/2$. $i.e., \quad L = \frac{\lambda}{2}$

Where, λ is the wavelength of incident suitable radiation and L is the length of the LASER cavity.

Carbon Dioxide Laser laser:

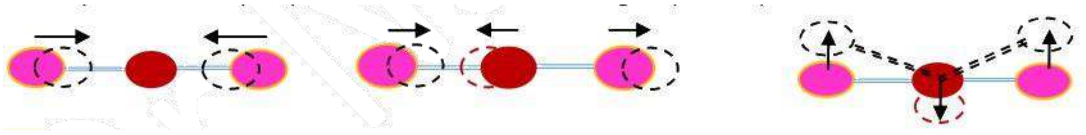
CO_2 laser invented by Kumar Patel in 1964. CO_2 is a linear, symmetric molecule with the carbon atoms balanced against the two oxygen atoms. O-C-O. Therefore, three characteristic vibrational modes exist.

A carbon dioxide molecule three independent modes of vibrations. They are

Symmetric: carbon atoms are at rest and both oxygen atoms vibrate simultaneously along the axis of the molecule departing or approaching the fixed carbon atoms. This mode is represented as (100) state (0.17 eV)

Asymmetric: oxygen atoms and carbon atoms vibrate asymmetrically, i.e., oxygen atoms move in one direction while carbon atoms in the other direction. This mode is represented as (001) state and it is highest among all (0.29 eV)

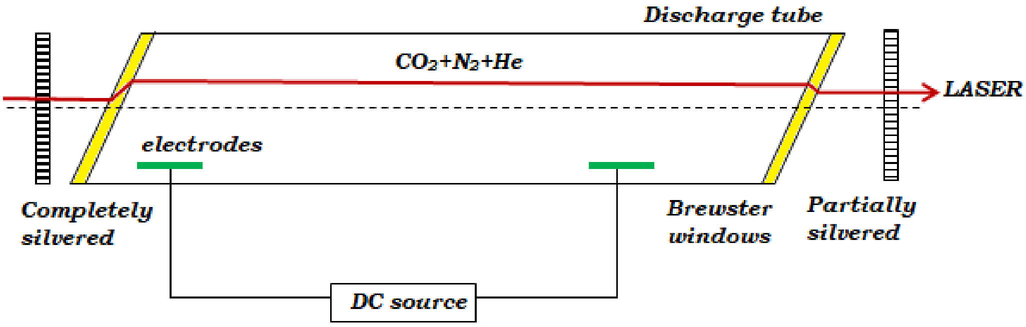
Bending: oxygen atoms and carbon atoms vibrate perpendicular to molecular axis. This mode is represented as (010) state and it is lowest among all (0.08 eV)



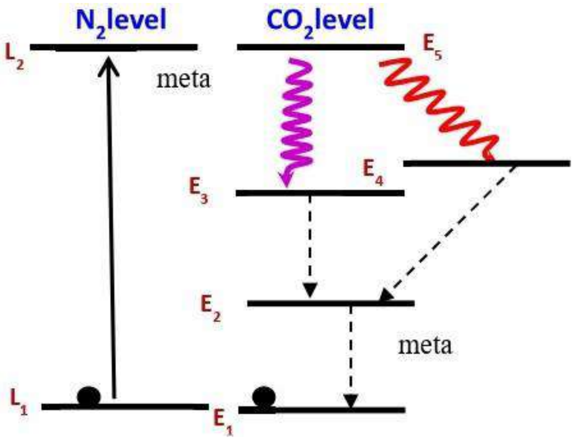
Principle : In N_2 molecule, the excited levels are metastable. The excited levels of N_2 coincidence in energy of CO_2 molecule. This helps in causing population inversion in CO_2 gas laser by means of transfer of energy.

Construction : It is a gas laser in which the gas mixture of CO_2, He & N_2 is filled in a discharge tube (radius 1cm, length 5m) at a pressure of 6-17 torr. The ratio of CO_2 :N_2 : He is 1:2:3.The mixture serves as **gain medium.** Electrical discharge is used as pumping method. The gas tube is fitted with Brewsters' windows in order to achieve plane polarization and two plane mirrors on either end. The whole system is water cooled.

(Brewster's law states that the tangent of the angle of polarization is numerically equal to the refractive index of the reflecting medium. i.e., $n = \tan\theta$)



Working:



The energy levels of N_2 and CO_2 are as shown in Fig. Lasing action takes place in both the levels. Hence both N_2 and CO_2 are considered as active medium

When high voltage is applied electric discharge occurs releasing electrons. Electrons collide with nitrogen molecules in the ground level exciting them to L_2 level. This is **first kind collision**

$$e+N_2 \rightarrow N_2*$$

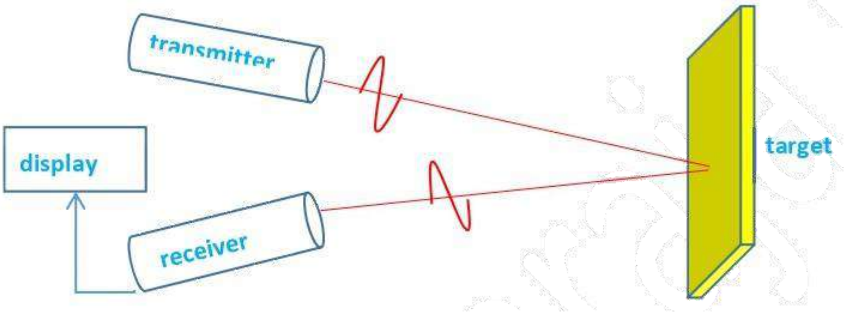
L_2 level is metastable and hence population inversion is achieved. The energy of L_2 is nearly same as E_5. Hence excited nitrogen molecules transfer their energy to CO_2 molecules in the ground state through collisions. This is **second kind collision**

$$N_2*+CO_2 \rightarrow CO_2*+N_2$$

Once all CO_2 are transferred to E_5 level, stimulated emission is triggered. Transition from E_5 to E_4 gives a beam of $\lambda= 10.6$ μm and transition from E_5 to E_3 gives a beam of $\lambda= 9.6$ μm. Afterwards CO_2 molecules return to E_1 level via radiative and non-radiation transitions.

Laser Range Finder (LRF)

LRF is a device used to measure distances more accurately and it works on the principle of a radar



A typical LRF consist of(i) transmitter, (ii) receiver, (iii) display and readout.

The transmitter uses a pulsed laser which sends out single, collimated and short pulse of laser radiation to the target. Target will reflect signals and these signals passes through filters. Filtered signal comes to receiver where it is amplified. The amplified signal enters the counter. By counting the number of pulses that have arrived, time of travel (t) can be calculated. Total distance travelled is $2d$ and time taken is t then

$$c = \frac{2d}{t}$$

$$or$$

$$d = \frac{ct}{2}$$

In ranging a target about 10 km away using these systems, accuracy within 5 m is easily obtained.

Problems:

(1) Show that the ratio of rate of spontaneous emission to induced absorption is given by $\left[1 - e^{-\frac{h\nu}{kT}}\right]$

$$\frac{\text{Rate of spontaneous emission}}{\text{Rate of induced absorption}} = \frac{A_{21}N_2}{B_{12}N_1U(\nu)}$$

$$But, \frac{N_2}{N_1} = e^{-\frac{h\nu}{kT}} \qquad \text{and} \qquad U_\nu = \frac{A_{21}}{B_{21}}\left[\frac{1}{e^{\frac{h\nu}{kT}} - 1}\right]$$

$$\therefore \frac{\text{Rate of spontaneous emission}}{\text{Rate of induced absorption}} = \frac{A_{21}}{B_{12}} e^{-\frac{h\nu}{kT}} \frac{1}{\frac{A_{21}}{B_{21}}\left[\frac{1}{e^{\frac{h\nu}{kT}} - 1}\right]}$$

$$Since \quad B_{12} = B_{21}$$

$$\frac{\text{Rate of spontaneous emission}}{\text{Rate of Induced absorption}} = e^{-\frac{h\nu}{kT}}\left[e^{\frac{h\nu}{kT}} - 1\right] = \left[1 - e^{-\frac{h\nu}{kT}}\right]$$

(2) Find the number of modes and their frequency separation in a resonant cavity of length 1 m of a laser operating at wavelength 632.8nm.

Soln.; L = 1m, λ = 632.8 nm = 632.8x10^{-9}m, n =?

$$WKT;\quad \lambda = \frac{2L}{n} \quad \text{OR} \quad n = \frac{2L}{\lambda}$$

$$n = \frac{2\times 1}{632.8\times 10^{-9}} = 3160556$$

$$Also,\quad \lambda = \frac{c}{\nu} = \frac{2L}{n} \quad \text{OR} \quad \nu = \frac{nc}{2L}$$

$$\therefore \nu_n = \frac{nc}{2L} \quad and \quad \nu_{n-1} = \frac{(n-1)c}{2L}$$

$$\therefore Frequency\ separation = \nu_n - \nu_{n-1}$$

$$\therefore Frequency\ separation = \frac{nc}{2L} - \frac{(n-1)c}{2L} = \frac{nc - nc + c}{2L} = \frac{c}{2L}$$

$$\therefore Frequency\ separation = \frac{c}{2L} = \frac{3\times 10^8}{2\times 1} = 1.5\times 10^8\, Hz$$

(3) A laser operating at 632.8nm emits 3.182x10^{16} photons per second. Calculate the output power of the laser. Also find the percentage power converted into coherent light energy, if the input power is 100 watt.

Soln.; λ = 632.8nm = 632.8x10^{-9}m, N = 3.182x10^{-16}photon/sec, p = ?, input power = 100 watt, %age power =?

$$WKT,\qquad N.\Delta E = E = Pt$$

$$\therefore N.\frac{hc}{\lambda} = Pt \quad \text{OR} \quad P = \frac{Nhc}{\lambda t}$$

$$P = \frac{3.182\times 10^{16}\times 6.626\times 10^{-34}\times 3\times 10^{8}}{632.8\times 10^{-9}\, x1} = 0.01\, watt$$

$$P = 0.01\, watt$$

$$Percentage\ Power = \frac{output\ power}{input\ power}\times 100 = \frac{0.01}{100}\times 100 = 0.01$$

$$Percentage\ Power = 0.01\%$$

(4) Calculate the wavelength of laser emitted from an extrinsic semiconductor laser if the band gap is 0.02eV. To which region of spectrum does it belong?

Soln.; λ = ?, E_g = 0.02 eV = 0.02x1.602x10^{-19} J

$$WKT,\quad E_g = h\nu = \frac{hc}{\lambda}$$

$$\therefore \lambda = \frac{hc}{E_g} = \frac{6.626\times 10^{-34}\times 3\times 10^{8}}{0.02\times 1.602\times 10^{-19}} = 6.208\times 10^{-5}\, m$$

This wavelength region belongs to infrared region (0.01 to 7x10^{-5}m).

NOTE: Visible region 7x10^{-5}m to 4x10^{-5}m, ultraviolet 4x10^{-5}m to 10^{-7}m

(5) A pulse from laser with power 1mW lost for 10nS, if the number of photons emitted per pulse is 3.491x10^{7}. Calculate the wavelength of laser. (May22)

Soln.; P= 1 x10^{-3}W, t=10 x10^{-9}sec, N=3.491x10^{7}, λ=?

$$N.\Delta E = Pt$$

$$N\frac{hc}{\lambda} = Pt \Rightarrow \lambda = \frac{Nhc}{Pt}$$

$$\therefore \lambda = \frac{3.491\times10^{7}\times6.625\times10^{-34}\times3\times10^{8}}{1\times10^{-3}\times10\times10^{-9}} = 6.938\times10^{-7}\,m$$

(6) A pulsed laser emits photons of wavelength 780nm with 20mW average power per pulse. Calculate the number of photons contained in each pulse if the pulse duration is 10ns.

Soln.; λ=780 x10^{-9}m, P= 20 x10^{-3}W, N=?, t=10 x10^{-9}sec

$$WKT, \quad N.\Delta E = Pt$$

$$N\frac{hc}{\lambda} = Pt \qquad \text{OR} \qquad N = \frac{\lambda Pt}{hc}$$

$$\therefore N = \frac{780\times10^{-9}\times20\times10^{-3}\times10\times10^{-9}}{6.625\times10^{-34}\times3\times10^{8}} = 7.849\times10^{8}\,photons$$

(6a) A laser source has a power output of 10^{-3}W. Calculate the number of photons emitted per second given wavelength of laser 692.8nm. (Model QP)

Soln., λ = 692.8 x10^{-9}m, P = 10^{-3}W, N = ?, t = 1sec

$$WKT, \quad N = \frac{\lambda Pt}{hc}$$

$$\therefore N = \frac{692.8\times10^{-9}\times10^{-3}\times1}{6.625\times10^{-34}\times3\times10^{8}} = 3.486\times10^{15}\,\text{photons/se cond}$$

(7) In a laser system, operating at 323K, the wavelength of the light emitted is 1.3μm. Determine the ratio of population of the energy levels (Boltzmann's factor).

Soln., T= 323K, λ=1.3 x10^{-6}m, $\frac{N_2}{N_1} = ?$

$$WKT, \quad \frac{N_2}{N_1} = e^{-\frac{h\nu}{KT}} = e^{-\frac{hc}{\lambda KT}}$$

$$\frac{N_2}{N_1} = e^{\frac{-6.626\times10^{-34}\times3\times10^{8}}{1.3\times10^{-6}\times1.38\times10^{-23}\times323}} = e^{-34.304} = 1.265x10^{-15}$$

(8) The ratio of population of two energy levels out of which one corresponds to meta stable state is 1.059 x 10^{-30}. Find the wavelength of light emitted at 330K.

Soln., $\frac{N_2}{N_1} = 1.059\times10^{-30}$, T = 330K, λ = ?

$$WKT, \quad \frac{N_2}{N_1} = e^{-\frac{h\nu}{KT}} = e^{-\frac{hc}{\lambda KT}}$$

$$OR \quad \ln\left(\frac{N_2}{N_1}\right) = \frac{-hc}{\lambda KT}$$

$$\therefore \lambda = \frac{-hc}{KT\ln\left(\frac{N_2}{N_1}\right)} = \frac{-6.626\times10^{-34}\times3\times10^{8}}{1.38\times10^{-23}\times330\times\ln\left(1.059\times10^{-30}\right)} = 6.323x10^{-7}\,m$$

$$OR \quad \lambda = 632.3nm$$

(8a) Calculate the ratio of population for a given pair of energy levels corresponding to emission of radiation 694.3nm at a temperature of 300K. (Model QP)

$$WKT, \quad \frac{N_2}{N_1} = e^{-\frac{h\nu}{KT}} = e^{-\frac{hc}{\lambda KT}}$$

$$\therefore \frac{N_2}{N_1} = e^{\frac{-6.626\times10^{-34}\times3\times10^{8}}{694.3\times10^{-9}\times1.38\times10^{-23}\times300}} = e^{-69.155} = 9.25\times10^{-31}$$

(9) Calculate the ratio of (i) Einstein's coefficients and (ii) stimulated to spontaneous emissions, for a system in thermal equilibrium at 300K in which radiations of wavelength 1.39µm are emitted.

Soln., $\frac{A_{21}}{B_{21}} = ?\frac{A_{21}}{B_{21}} = ?$ and $\frac{B_{12}}{A_{21}} = ?$, Rate of stimulated to spontaneous=? λ=1.39 X10^{-6}m, T=300K

$$(i)\, WKT, \quad \frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3} = \frac{8\pi h}{\lambda^3} = \frac{8\pi\times6.626\times10^{-34}}{\left(1.39\times10^{-6}\right)^3} = 6.2\times10^{-15}$$

$$\frac{A_{21}}{B_{21}} = 6.2\times10^{-15}$$

$$(ii)\, WKT, \frac{\text{Rate of stimulated emission}}{\text{Rate of spontaneous emission}} = \frac{B_{21}N_2U(\nu)}{A_{21}N_2}$$

$$But, \ \mathrm{U}_\nu = \frac{A_{21}}{B_{21}}\left[\frac{1}{e^{\frac{h\nu}{kT}}-1}\right]$$

$$\therefore \frac{\text{Rate of stimulated emission}}{\text{Rate of spontaneous emission}} = \frac{B_{21}}{A_{21}}.\frac{A_{21}}{B_{21}}\frac{1}{\left(e^{\frac{hc}{\lambda kT}}-1\right)} = \frac{1}{\left(e^{\frac{hc}{\lambda kT}}-1\right)}$$

$$\therefore \frac{\text{Rate of stimulated emission}}{\text{Rate of spontaneous emission}} = \frac{1}{\left(e^{\frac{6.626\times10^{-34}\times3\times10^{8}}{1.39\times10^{-6}\times1.38\times10^{-23}\times300}}-1\right)} = \frac{1}{\left(e^{34.543}-1\right)} = \frac{1}{\left(1.004x10^{15}-1\right)} = 9.96\times10^{-16}$$

Model Questions:

1. What is LASER? Enumerate the Characteristics of a LASER Beam.
2. Discuss the three possible ways through which radiation and matter interaction can take place.
3. Explain the terms, (i) Induced absorption, (ii) Spontaneous emission, (iii) Stimulated emission, (iv) Population inversion, (v) Meta-stable state & (vi) Resonant cavity.
4. Explain the rates of absorption and emission and hence derive an expression for energy density using Einstein's A and B coefficients.
5. Explain requisites of LASER system.
6. What is Semiconductor LASER? Describe with energy band diagram the construction & working of Semiconductor diode LASER along with applications.
7. Discuss the working of LASER barcode reader.
8. With the help of a sketch describe the principle, construction and working of CO_2 Laser.
9. Explain LASER Range Finder in defence as application.

Numerical Problems:

1. Find the ratio of population of two energy levels in a LASER if the transition between them produces light of wavelength 6493 Å, assuming the ambient temperature at 27°C.
2. Find the ratio of population of two energy levels in a medium at thermal equilibrium, if the wavelength of light emitted at 291 K is 6928 Å.
3. The ratio of population of two energy levels out of which one corresponds to metastable state is 1.059×10^{-30}. Find the wavelength of light emitted at 330 K.
4. Find the ratio of population of two energy levels in a medium at thermal equilibrium, if the wavelength of light emitted at 300 K is 10μm. Also find the effective temperature when energy levels are equally populated.
5. The average power output of a LASER beam of wave length 6500 Å is 10 mW. Find the number of photons emitted per second by the LASER source.
6. The average power of a LASER beam of wavelength 6328 Å is 5 mW. Find the number of photons emitted per second by the LASER source.
7. A pulsed LASER has an average power output 1.5 mW per pulse and pulse duration is 20 ns. The number of photons emitted per pulse is estimated to be 1.047×10^{8}. Find the wavelength of the emitted LASER.
8. A pulsed LASER with power 1 mW lasts for 10 ns. If the number of photons emitted per pulse is 5×10^{7}. Calculate the wavelength of LASER.

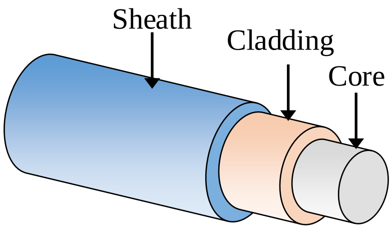
9. A Ruby LASER emits a pulse of 20 ns duration with average power per pulse being 100 kW. If the number of photons in each pulse is 6.981×10^{15}, calculate the wavelength of photons.
10. In a LASER system when the energy difference between two energy levels is 2×10^{-19} J, the average power output of LASER beam is found to be 4 mW. Calculate number of photons emitted per second.

OPTICAL FIBERS

Introduction

Optical fiber is a device used to transmit light through bundle of thin fibers of transparent dielectric material from one end to another end for a very long distance. It works on the principle of Total Internal Reflection (TIR).

Construction: The sectional view of a typical optical fiber is as shown in the figure. It has three regions named as Core, Cladding and Sheath.



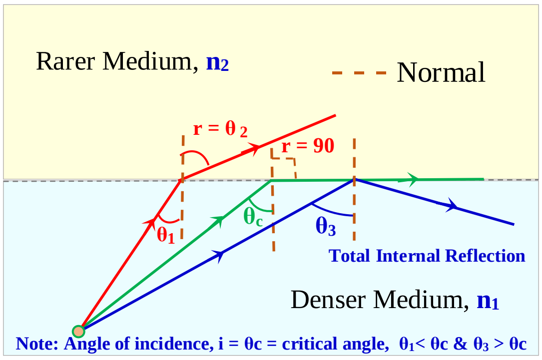
Optical Fiber

1. The innermost light guiding region is called Core.
2. The layer covering core is called Cladding or Clad, which helps in total internal reflection of light.
3. The outermost protective layer is called Sheath (Coating), which protects the fiber from mechanical stress and chemical reactions.

The optical fiber is designed to support total internal reflection and hence the refractive index (RI) of core n_1 is made greater than the RI of cladding n_2. A typical fiber will be of the order of few microns.

Total Internal Reflection

Consider a ray of light moving from a denser medium of refractive index n_1 to rarer medium of refractive index n_2. As a result, the incident ray of light bends away from the normal. Hence the angle of refraction θ_2 is greater than the angle of incidence θ_1. As the angle of incidence increases the angle of refraction also increases. For a particular angle of incidence, $i = \theta_c$ the refracted ray grazes the interface separating the two media. The corresponding angle of incidence θ_c is called **critical angle**. If the angle of incidence is greater than the critical angle θ_c, then the light ray is turned back into the same medium and is called **T**otal **I**nternal **R**eflection.



The above figure shows Total Internal Reflection;

According to Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

when $\theta_1 = \theta_c$, then $\theta_2 = 90^\circ$ and $\sin 90^\circ = 1$

$$\therefore \; n_1 \sin\theta_c = n_2$$

$$\therefore \; \sin\theta_c = \frac{n_2}{n_1} \quad \text{or} \quad \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

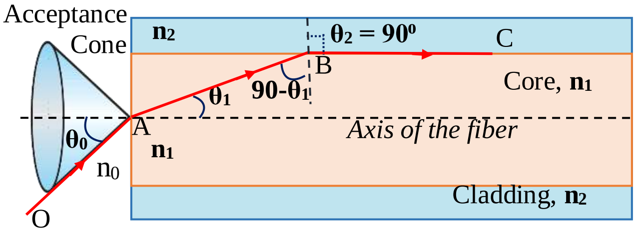
Angle of acceptance and Numerical aperture (NA):

Acceptance angle (θ) is the maximum angle of incidence with which the ray is sent into the fiber core which allows the incident light to be guided by the core. It is also called as waveguide acceptance angle or acceptance cone half angle.

In optics, the numerical aperture (NA) of an optical fiber is a dimensionless number that characterizes the range of angles over which the fiber can accept light. Numerical aperture represents the light gathering capability of optical fiber and it is defined as sine of the acceptance angle.

$$\therefore \ \mathrm{NA} = \sin\theta_0$$

Condition for propagation (Derivation for Angle of Acceptance)



Consider an optical fiber with core made of refractive index n_1 and cladding made of refractive index n_2. Let n_0 be the refractive index of the surrounding medium.

Let a ray of light OA entering into the core at an angle of incidence θ_0 w.r.t fiber axis. Then it is refracted along AB at an angle θ_1 and meet the core-cladding interface at critical angle of incidence ($\theta_c = 90 - \theta_1$). Then the refracted ray grazes along BC.

By applying Snell's law at A, we get

$$n_0 \sin\theta_0 = n_1 \sin\theta_1$$

$$\sin\theta_0 = \frac{n_1}{n_0}\sin\theta_1 \qquad (1)$$

Again by applying Snell's law at B, we get

$$n_1 \sin(90-\theta_1) = n_2 \sin 90^\circ \qquad \text{since, } \sin(90-\theta_1) = \cos\theta_1 \text{ and } \sin 90^\circ = 1, \text{ we get}$$

$$n_1 \cos\theta_1 = n_2 \quad \text{or} \quad \cos\theta_1 = \frac{n_2}{n_1} \qquad (2)$$

$$\sin^2\theta_1 + \cos^2\theta_1 = 1 \quad \text{or} \quad \sin\theta_1 = \sqrt{1-\cos^2\theta_1}$$

$$\therefore \sin\theta_1 = \sqrt{1-\frac{n_2^2}{n_1^2}} \qquad (3)$$

By substituting this in equation (1), we get

$$\sin\theta_0 = \frac{n_1}{n_0}\sqrt{1-\frac{n_2^2}{n_1^2}} = \frac{n_1}{n_0}\frac{1}{n_1}\sqrt{n_1^2-n_2^2} = \frac{1}{n_0}\sqrt{n_1^2-n_2^2} \qquad (4)$$

Since, Numerical Aperture, NA = $\sin\theta_0$,

$$\therefore NA = \frac{1}{n_0}\sqrt{n_1^2-n_2^2}$$

If the surrounding medium is air, then $n_0 = 1$

$$\therefore NA = \sin\theta_0 = \sqrt{n_1^2 - n_2^2} \qquad (5)$$

Light is transmitted through the fiber only when

$$\therefore \theta_i \le \theta_0 \quad \text{or} \quad \sin\theta_i \le \sin\theta_0 \le \sqrt{n_1^2 - n_2^2}$$

$$\therefore \sin\theta_i \le NA \le \sqrt{n_1^2 - n_2^2} \qquad (6)$$

This is the condition for propagation for light through the optical fiber with multiple total internal reflections.

Fractional index change (Δ):

It is the ratio of the difference between the refractive indices of the core and cladding to the refractive index of core of an optical fiber.

$$\text{i.e., } \Delta = \frac{n_1 - n_2}{n_1}$$

Δ is always positive and less than 1. Because, $n_1 > n_2$.

Relation between NA and Δ:

$$\text{WKT, } NA = \sqrt{n_1^2 - n_2^2} \quad \text{and} \quad \Delta = \frac{n_1 - n_2}{n_1}$$

$$NA = \sqrt{(n_1 - n_2)(n_1 + n_2)}$$

$$NA = \sqrt{\Delta n_1 (n_1 + n_2)}$$

$$\text{Since } n_1 \approx n_2,\ NA = \sqrt{\Delta n_1 (2n_1)}$$

$$NA = n_1\sqrt{2\Delta}$$

Modes of Propagation: In an optical fiber the wave propagation mode is referred to as fiber modes. The light ray paths along which the waves are in phase inside the fiber are known as modes.

In simple words, the allowed paths for the light ray inside the fiber are known as modes of propagation.

$$\textbf{Number of modes} \propto \frac{\mathbf{d}}{\boldsymbol{\lambda}}$$

Where, d is the diameter of the core and L is the wavelength of the light ray travelling through the fiber.

The different types of fiber modes are guided mode, leaky mode and radiation mo de. In an optical communication system, signals are transmitted using guided modes only.

V- Number: The number of modes supported for propagation in the fiber depends on the core size, operating wavelength and refractive indices of core and cladding materials. It can be conveniently defined using a normalized frequency parameter, called V- number and it is given by

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2}$$

Where λ is the (operating) wavelength of light propagating in the fiber.

d is the core diameter

n_1 is the refractive index of the core.

n_2 is the refractive index of the cladding.

For V>>1, the number of modes supported by the fiber is given by

$$N = \frac{V^2}{2}$$

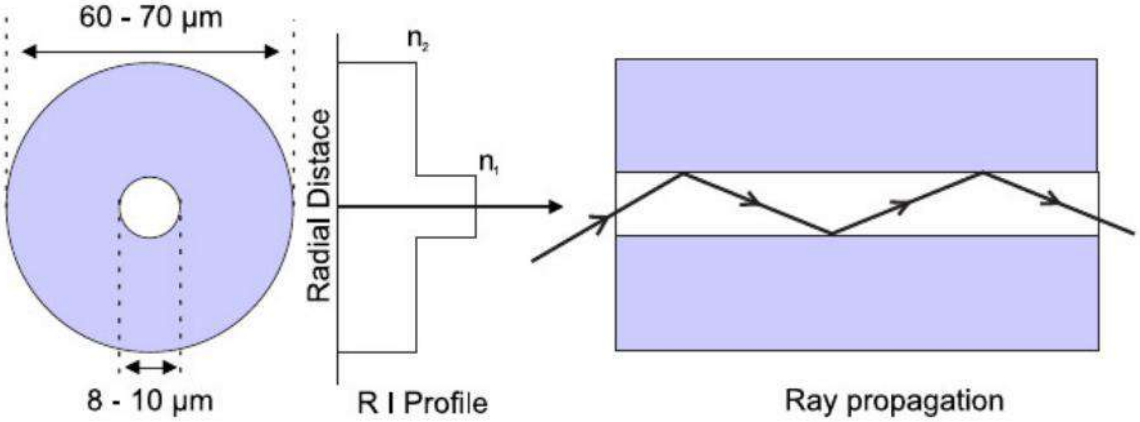
Refractive index profile: Refractive index profile is the distribution of refractive indices of materials within an optical fiber. Some optical fiber has a step-index profile, in which the core has one uniformly-distributed index and the cladding has a lower uniformly-distributed index. Other optical fiber has a graded-index profile, in which the refractive index varies gradually as a function of radial distance from the fiber center.

Types of optical fiber:

Based on the refractive index profile, core size and mode of propagation, the fibers are classified into three types, namely

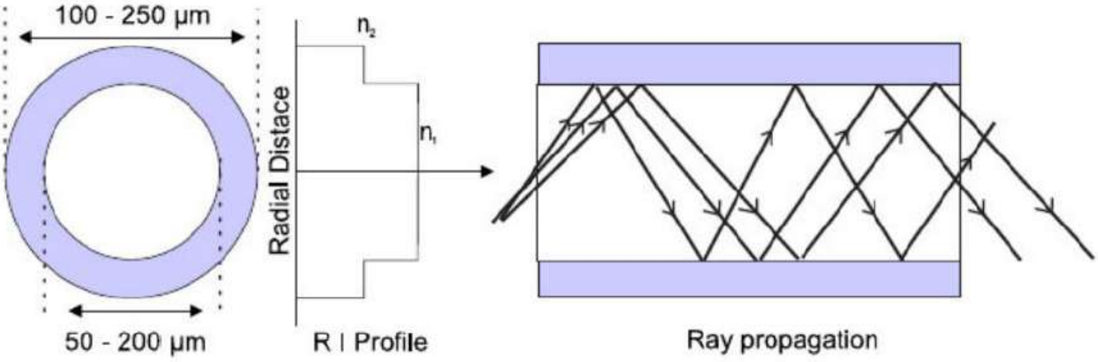
1. Step-index single mode fiber
2. Step-index multi-mode fiber
3. Graded-index multi-mode fiber

1. Step-index single mode fiber:



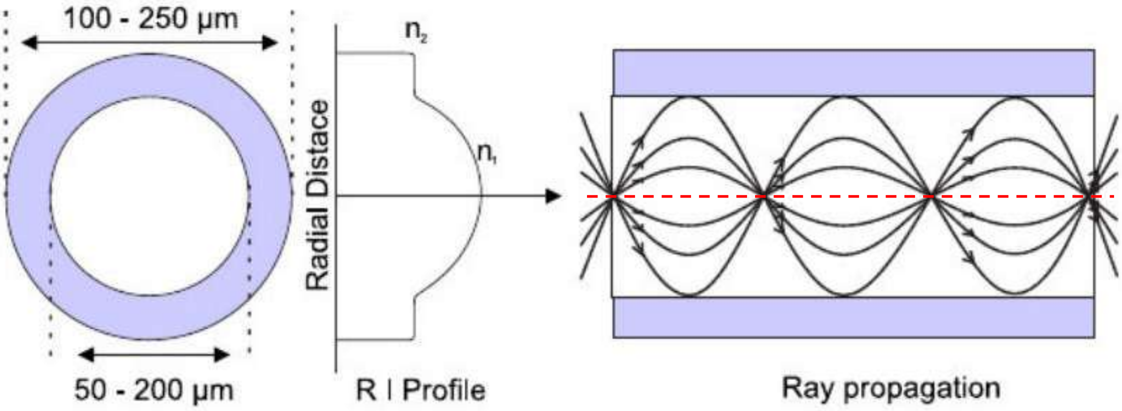
A single mode step index fiber consists of a very fine thin core (made of glass material) of uniform refractive index n_1 is surrounded by a cladding of refractive index n_2 lower than that of the core, Since there is abrupt change in RI of core and cladding at the interface it is called step index fiber. The diameter of the core is about 8 to 10 μm and that of cladding is about 60 to 70 μm. Since the core size is small, the numerical aperture is also small and hence it supports for single mode as shown in the figure. This need laser as the source of light.

2. Step-index multi-mode fiber:



Step-index multimode fiber is similar to that of a single mode fiber, but it has a larger core diameter and constant refractive index, by the virtue of which it will be able to support for large number of modes of propagation as shown in figure. The diameter of the core is about 50 to 200 μm and that of the cladding is about 100 to 250 μm. The step-index multi mode fiber can accept either a laser or LED as source of light. It is the least expensive of all. They are used in data links.

3. Graded-index multi-mode fiber:



A graded multimode fiber has concentric layers of RI is called GRIN fiber. That means the RI of the core varies with distance from the axis of the fiber. The refractive index of the core along the axis of the fiber is maximum and it decreases uniformly on either side of the axis towards the core-cladding interface. Hence the refractive index profile follows a parabolic shape and light transmissions in the fiber are shown in figure. The diameter of the core and that of cladding are almost same as that of multi-mode index fibers. Either a laser or LED's are used as light sources for operating the system. It is the most expensive of all and used in telecommunication links.

Attenuation (Transmission loss or fiber loss):

Attenuation is the reduction in power or intensity of light as it travels in the fiber. The reduction may be due to light absorption, scattering and radiation losses (extensive fiber bends). The net attenuation is given by a factor called the attenuation coefficient (α) in dB/km and it is defined as optical power output to the optical power input for a fiber of length L and a wavelength of the propagating light.

$$\alpha = \frac{-10}{L}\log\left(\frac{P_{out}}{P_{in}}\right) \quad dB/Km$$

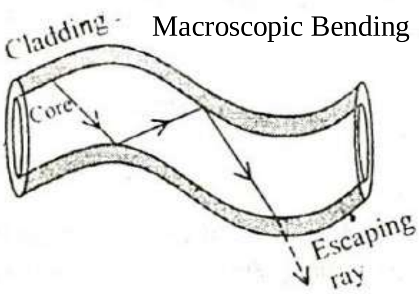
Attenuation in an optical fiber is comparatively less than that in coaxial cables.

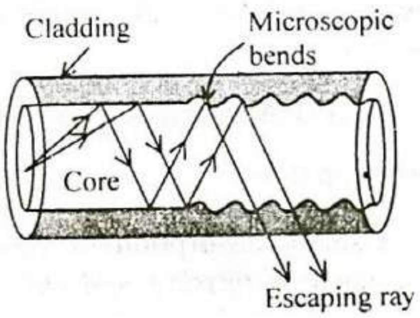
Causes of attenuation: The three mechanism through which attenuation takes place are

(1) Absorption losses: The absorption losses occur due to the presence of impurities (like Cr, Cu, ions trapped within the glass at the time of manufacture) or due to the basic material (glass) itself absorb energy at certain wavelength. Typical absorption losses are of the order of 0.1dB/km in the 0.8 to 1.6 µm wavelength ranges and 0.03dB/km in the 1.3 to 1.6 µm wavelength range.

(2) Scattering losses: Scattering losses occur due to imperfections and impurities in the fiber material. Refractive index changes while the signal travels in the fiber. This sharp variation in refractive index is induced by the localized structural inhomogeneity. This type of scattering is same as Rayleigh scattering. Rayleigh scattering occurs whenever a light wave travels through a medium having scattering objects whose dimensions are smaller than a wavelength. Thus it becomes a loss.

(3) Geometrical loss (Radiation loss):





Geometrical losses occur due to (a) macroscopic and (b) microscopic bends.

(a) Due to sharp bends, some of the light energy escapes through the cladding and leads to loss in the intensity of light ray.

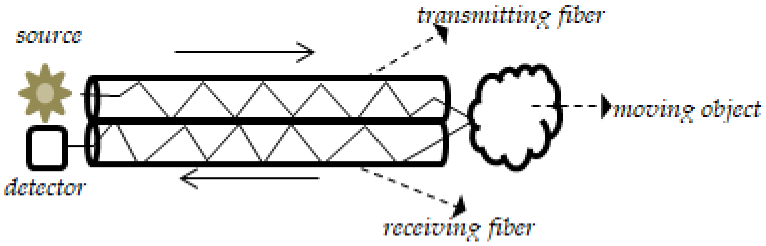
(b) The microscopic bends cause irregular reflections and some of them then leak through the fiber.

Applications:

1. Intensity Modulated Optical Fiber Displacement Sensor :

It consists of a bundle of transmitting fibers coupled to the laser source and a bundle of receiving fibers coupled to the detector as shown in the figure.

The axis of the transmitting fiber and the receiving fiber with respect to the moving target can be adjusted to increase the sensitivity of the sensor.

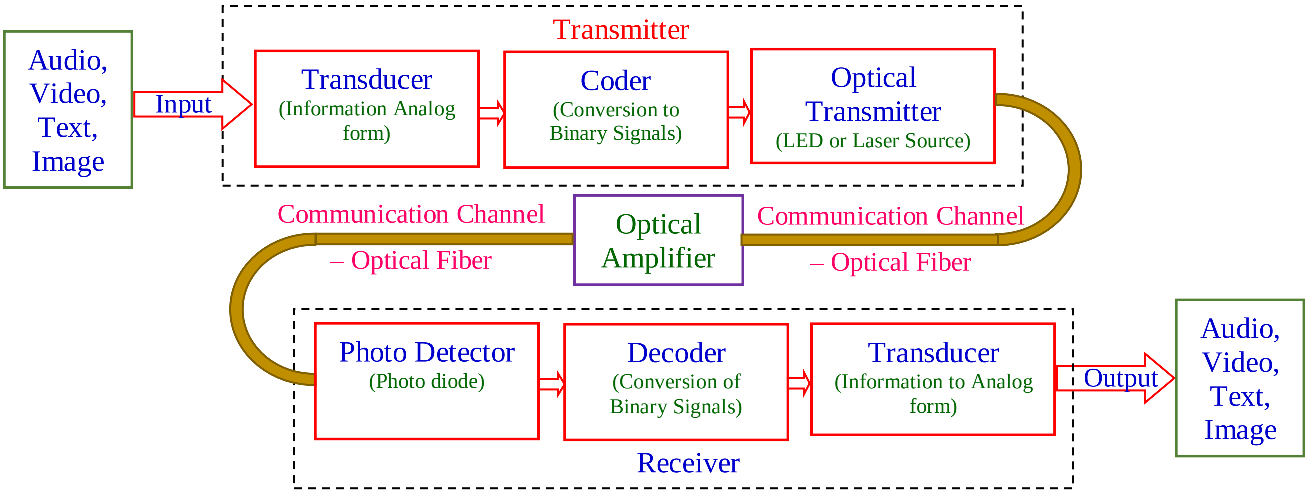


Light from the source is transmitted through the transmitting fiber and is made to fall on the moving target. The light reflected from the target is made to pass through the receiving fiber and the same is detected by the detector.

Based on the intensity of the light received, the displacement of the target can be measured, (i.e.) if the received intensity is more then we can say that the target is moving towards the sensor and if the intensity is less, we can say that the target is moving away from the sensor.

(2) Fiber Optic communication:

Block Diagram



A basic of point to point communication system using optical fiber is shown in figure. The voice of a telephone user gives rise to electrical signals. These electrical signals are converted to binary data using coder. These electrical pulses are converted into pulses of optical power by an optical source (such as an LED or laser) in the binary form.

Now the light pulses is coupled into the optical fiber at an incidence angle less than that of the acceptance angle. The light pulses inside the fiber undergo total internal reflection and reach the other end of the fiber and fed into a light detector. Light detector converts the light signals into pulses of electric signals (current). These pulses are further decoded into analog electrical signal and converted into the usable form like audio or video.

Advantages (Merits) of optical communication system:

1) Transmission loss (attenuation) is low.
2) Fiber is lighter and compact than equivalent copper cables.
3) Fiber have large data rate compared to equivalent copper cables.
4) There is no interference in the transmission of light from electromagnetic waves generated by electrical appliances.
5) Fibers are free from corrosion effect caused by salt, pollution and radiation.
6) Cost of the fiber is less compared to the other modes of communication.

Demerits (Disadvantages) :

1) The joining of two ends of the separated fiber called splicing, is too difficult and fiber loss is more.
2) Repeaters are required at regular interval of lengths to amplify the weak signal in long distance communication.
3) Bends will increase the loss of the fiber.
4) Maintenance cost of the systems (Repeaters, detectors etc.,) with optical fibers are very high.
5) Fibers undergo expansion and contraction with temperature that upset some critical alignment which lead to loss in signal power.

Problems:

(1) Calculate the numerical aperture and angle of acceptance of a given optical fiber if the refractive index of the core and cladding are 1.563 and 1.498 respectively.

Soln.: NA = ?, θ_a = ? $n_1 = 1.563$ & $n_2 = 1.498$

$$NA = \sqrt{n_1^2 - n_2^2}$$

$$NA = \sqrt{(1.563)^2 - (1.498)^2}$$

$$NA = 0.446$$

$$Sin\,\theta_a = NA$$

$$\theta_a = Sin^{-1}(0.446)$$

$$\theta_a = 26.5^0$$

(2) An optic glass fiber of refractive index 1.450 is to be clad with another glass to ensure internal reflection that will contain light traveling within 5 degree of the fiber axis. What maximum index of refraction is allowed for the cladding?

Soln.: $n_1 = 1.450$, $\theta_r = 5^0$ (or i = 90-5 = 85^0), $n_2 = ?$

$$\cos\theta_r = \frac{n_2}{n_1} \qquad \text{or} \qquad n_2 = n_1\cos\theta_r$$

$$\therefore n_2 = 1.45\cos 5 = 1.444$$

(3) An optic fiber has a NA of 0.2 and a cladding refractive index of 1.59. Determine the core refractive index and also the acceptance angle for the fiber in water which has a refractive index of 1.33

Soln.: NA=0.2, n_2=1.59, n_1=? , θ_a=? , n_0=1.33

$$Sin\,\theta_a = NA$$

$$\therefore\ \theta_a = Sin^{-1}(0.2) = 11.54^0 = 11^\circ 32'$$

$$NA = \sqrt{\frac{n_1^2 - n_2^2}{n_0^2}}$$

$$n_1 = \sqrt{(NA)n_0^2 + n_2^2} = 1.612$$

(4) An optical has core refractive index 1.5 and clad refractive index 3% less than that of core. Calculate NA, angle of acceptance and internal critical angle.

Soln.: $n_1 = 1.5$, $n_2 = 1.5$ - (3% of 1.5), NA = ?, θ_a = ? & θ_c = ?

n_2 =1.5 - (0.03 x 1.5) =1.455

$$NA = \sqrt{n_1^2 - n_2^2} = \sqrt{(1.5)^2 - (1.455)^2} = 0.3647$$

$$\theta_a = Sin^{-1}(NA) = Sin^{-1}(0.3647) = 21.39^\circ$$

$$\theta_a = Sin^{-1}\left(\frac{n_2}{n_1}\right) = Sin^{-1}\left(\frac{1.455}{1.5}\right) = 75.93^\circ$$

(5) (Isem 2023) The angle of accepatance of an optical fibre is 30^0 when kept in air. Find the angle of accptance when it is in medium of refractive index 1.33

Soln.: Soln.: $\theta_a = 30^\circ$, $n_0 = 1.33$, $\theta'_a = ?$

$$Sin\,\theta_a = \sqrt{n_1^2 - n_2^2} \quad \text{and} \quad Sin\,\theta_a^1 = \sqrt{\frac{n_1^2 - n_2^2}{n_0^2}}$$

$$\therefore \frac{Sin\theta_a^1}{Sin\theta_a} = \sqrt{\frac{n_1^2 - n_2^2}{n_0^2}} \frac{1}{\sqrt{n_1^2 - n_2^2}}$$

$$\frac{Sin\theta_a^1}{Sin\theta_a} = \frac{1}{n_0} \quad \text{or} \quad Sin\theta_a^1 = \frac{Sin\theta_a}{n_0}$$

$$Sin\theta_a^1 = \frac{Sin30}{1.33} \quad \text{or} \quad \theta_a^1 = Sin^{-1}[0.3759]$$

$$\theta_a^1 = 22.08^\circ$$

(6) A fiber sample 500 m long has an input power of 8.6 micro watt and an output power of 7.5 µW. What is the loss specification for the cable sample?

Soln.: L = 500 m = 0.5 km, P_{in}= 8.6 µW, P_{out} = 7.5 µW, $\alpha = ?$

$$\alpha = \frac{-10}{L}\log\left(\frac{P_{out}}{P_{in}}\right) dB/km$$

$$\alpha = \frac{-10}{0.5}\log\left(\frac{7.5}{8.6}\right) = 1.19\ dB/km,$$

(7) The attenuation of light in an optical fiber is estimated to be 2.0dB/km. What fraction of the initial intensity remains after 1 km and after 8 km?

Soln.: $\propto$ = 2.0 dB/km, $\left(\frac{P_{out}}{P_{in}}\right)_{1Km} = ?\ \left(\frac{P_{out}}{P_{in}}\right)_{8Km} = ?$

$$\alpha = \frac{-10}{L}\log\left(\frac{P_{out}}{P_{in}}\right) dB/km \quad \text{or} \quad \left(\frac{P_{out}}{P_{in}}\right) = 10^{\left(\frac{-\alpha L}{10}\right)}$$

Intensity remaining after 1 km is

$$\left(\frac{P_{out}}{P_{in}}\right)_{1km} = 10^{\left(\frac{-2\times 1}{10}\right)} = 0.631$$

Intensity remaining after 8 km is

$$\left(\frac{P_{out}}{P_{in}}\right)_{8km} = 10^{\left(\frac{-2\times 8}{10}\right)} = 0.025$$

(8) An optical signal propagating in a fiber retains 85% of input power after traveling a distance of 500 m in the fiber. Calculate the attenuation coefficient.

Soln.:

$$\alpha = \frac{-10}{L}\log\left(\frac{P_{out}}{P_{in}}\right) dB/km$$

$$\alpha = \frac{-10}{L}\log\left(\frac{85}{100}\right) = 1.412\ dB/km$$

(8) In a step index fiber, the relative refractive index difference is 2% and refractive index of cladding is 1.4. Calculate the refractive index of core and also the critical propagation angle.

Soln.: Δ=2%=0.02, n_2=1.4, n_1=?, θ_r=?

$$\Delta = \frac{n_1 - n_2}{n_1} = 1 - \frac{n_2}{n_1}$$

$$or \quad \frac{n_2}{n_1} = 1 - \Delta \quad or \quad n_1 = \frac{n_2}{1-\Delta}$$

$$\therefore \quad n_1 = \frac{1.4}{1-0.02} = 1.4285$$

$$\cos\theta_r = \frac{n_2}{n_1} = \frac{1.4}{1.4285} = 0.98$$

$$\theta_r = \cos^{-1}(0.98) = 11.48 = 11°29'$$

(9) Consider a slab waveguide made of AlGaAs having RI for core and clad 3.6 and 3.55 respectively. Find, how many modes can propagate in this waveguide if d = 5λ

Soln.: n_1=3.6, n2=3.55, N=?, d=5λ

$$N = \frac{V^2}{2} = \frac{1}{2}\left[\frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2}\right]^2$$

$$N = \frac{1}{2}\left[\frac{\pi \times 5\lambda}{\lambda}\sqrt{(3.6)^2 - (3.55)^2}\right]^2$$

$$N = 44.10 \approx 44$$

(10) A step- index optical fiber has a core index of 1.46 and the cladding index of 1.409. If the core diameter is 80micro m and the wavelength of the light source is 1.2micro m, determine the number of modes present in the fiber.

Soln.: $n_1 = 1.46$, $n_2 = 1.409$, d = 80×10^{-6} m, λ = 1.2×10^{-6} m, N = ?

$$N = \frac{1}{2}\left[\frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2}\right]^2$$

$$N = \frac{1}{2}\left[\frac{\pi \times 80 \times 10^{-6}}{1.2 \times 10^{-6}}\sqrt{(1.46)^2 - (1.409)^2}\right]^2$$

$$N = 3209.13 \approx 3209$$

(11) A single mode step index optical fiber used in communication has a core with refractive index 1.45, refractive index change of 5x10-3 and a core diameter of 6μm. If the operating wavelength of the communication system is 1.2 micro m, determine the V- parameter of the cable.

Soln.: n_1=1.45, $\Delta = 5\times10^{-3}$, d = 6μm = 6×10^{-6}m, λ = 1.2×10^{-6} m, V = ?

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2}$$

$$V = \frac{\pi d}{\lambda} n_1 \sqrt{2\Delta}$$

$$V = \frac{\pi \times 6\times 10^{-6}}{1.2\times 10^{-6}} \times 1.45 \times \sqrt{2\times 5\times 10^{-3}}$$

$$V = 2.28$$

(12) (May22) Calculate NA, Relative RI, V- number and the number of modes in an optical fiber of core diameter 50µm and core and cladding RI are 1.41 and 1.40 respectively. Given wavelength of source 820 nm.

Soln.: NA = ?, Δ = ?, V = ?, N = ?, d = 50 µm =50×10^{-6} m, n_1 = 1.41, n_2 = 1.40, λ = 820×10^{-9} m

$$NA = \sqrt{n_1^2 - n_2^2} = \sqrt{(1.41)^2 - (1.40)^2} = 0.168$$

$$\Delta = \frac{n_1 - n_2}{n_1} = \frac{1.41 - 1.40}{1.41} = 7.09\times 10^{-3}$$

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2} = \frac{\pi \times 50\times 10^{-6} \times \sqrt{(1.41)^2 - (1.40)^2}}{820\times 10^{-9}} = 32.11 \qquad N = \frac{V^2}{2} = \frac{(32.11)^2}{2} = 515$$

(13) The refractive indices of the core and cladding of a step- index optical fiber are 1.45 and 1.40 respectively and its core diameter is 45µm. Calculate its relative refractive index difference, NA, V- number at wavelength 1000nm and the number of modes.

Soln.: n_1 = 1.45, n_2 = 1.40, d = 45 µm = 45×10^{-6} m, λ=1000×10^{-9} m, Δ = ?, NA = ?, V = ?, N =?

$$\Delta = \frac{n_1 - n_2}{n_1} = \frac{1.45 - 1.40}{1.45} = 0.0345$$

$$NA = \sqrt{n_1^2 - n_2^2} = \sqrt{(1.45)^2 - (1.40)^2} = 0.38$$

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2} = \frac{\pi \times 45\times 10^{-6} \times 0.38}{1000\times 10^{-9}} = 53.72$$

$$N = \frac{V^2}{2} = \frac{(53.72)^2}{2} = 1443$$

Model Questions:

1. Define the terms: (i) angle of acceptance, (ii) numerical aperture, (iii) modes of propagation & (iv) refractive index profile.
2. Obtain an expression for numerical aperture and arrive at the condition for propagation.
3. Explain modes of propagation and RI profile.
4. What is attenuation? Explain the factors contributing to the fiber loss.
5. Discuss the types of optical fibers based on modes of propagation and RI profile.

6. Explain attenuation along with the expression for attenuation co-efficient and also discuss the types of fiber losses.
7. Discuss point to point optical fiber communication system and mention its advantages over the conventional communication system.
8. Discuss the advantages and disadvantages of an optical communication.

Numerical Problems:

1. Calculate the numerical aperture and angle of acceptance for an optical fiber having refractive indices 1.563 and 1.498 for core and cladding respectively.
2. The refractive indices of the core and cladding of a step index optical fiber are 1.45 and 1.4 respectively and its core diameter is 45 µm. Calculate its fractional refractive index change and numerical aperture.
3. Calculate numerical aperture, acceptance angle and critical angle of a fiber having a core RI 1.50 and cladding RI 1.45.
4. An optical fiber has a numerical aperture of 0.32. The refractive index of cladding is 1.48. Calculate the refractive index of the core, the acceptance angle of the fiber and the fractional index change.
5. An optical signal propagating in a fiber retains 85% of input power after travelling a distance of 500 m in the fiber. Calculate the attenuation coefficient.
6. An optical fiber has core RI 1.5 and RI of cladding is 3% less than the core index. Calculate the numerical aperture, angle of acceptance critical angle.
7. The numerical aperture of an optical fiber is 0.2 when surround by air. Determine the RI of its core, given the RI of the cladding is 1.59. Also find the acceptance angle when the fiber is in water of RI 1.33.
8. The angle of acceptance of an optical fiber is 300 when kept in air. Find the angle of acceptance when it is in medium of refractive index 1.33.
9. An optical fiber of 600 m long has input power of 120 mW which emerges out with power of 90 mW. Find attenuation in fiber.
10. The attenuation of light in an optical fiber is 3.6 dB/km. What fraction of its initial intensity is remains after i) 1 km and ii) 3 km ?
11. The attenuation of light in an optical fiber is 2.2 dB/km. What fraction of its initial intensity is remains after i) 2 km and ii) 6 km?

* * * END * * *

MYSURU ROYAL INSTITUTE OF TECHNOLOGY

DEPARTMENT OF PHYSICS

Applied Physics for EEE Stream : 2022-23

Module – 4: Maxwell's Equations and EM waves

Syllabus:

Maxwell's Equations and EM waves: 8 Hrs

Maxwell's Equations: Fundamentals of Vector Calculus. Divergence and Curl of Electric field and Magnetic field (static), Gauss' divergence theorem and Stoke's theorem. Description of laws of Electrostatics, Magnetism, Faraday's laws of EMI, Current Density, Equation of Continuity, Displacement Current (with derivation), Maxwell's equations in vacuum, Numerical Problems

EM Waves: The wave equation in differential form in free space (Derivation of the equation using Maxwell's equations), Plane Electromagnetic Waves in vacuum, their transverse nature.

Pre-requisite: Electricity & Magnetism

Self-learning: Fundamentals of vector calculus.

Module-4

Maxwell's Equations and EM waves

Vector Analysis

Introduction

We come across with various types of physical quantities in our daily life. They are classified into two categories *viz* **Scalars** and **Vectors.**

Scalars

Any physical quantity having only magnitude is known as *Scalar*. Just a number along with an appropriate unit determines it. There is no scope for the direction. Time, temperature, Speed, energy, work done, voltage, charge, power, volume, etc are some more examples for scalar.

Vectors

Vector is a quantity having both magnitude and direction. Displacement, velocity, acceleration, force, electric field intensity, momentum, torque, etc are vectors.
A vector is represented by as follows

$$\vec{R} = |\vec{R}|\hat{a}$$

Where, $|\vec{R}|$ = magnitude of the vector

$\hat{a}$ = direction of the vector

The magnitude of $\vec{R}$ is given in terms of its coordinates as

$$|\vec{R}| = \sqrt{x^2 + y^2 + z^2}$$

Unit vector is given by

$$\hat{a} = \frac{\vec{R}}{|\vec{R}|} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Here i, j, k are unit vectors
Now consider two distinct points P (x_1,y_1,z_1) and Q (x_2,y_2,z_2) then

$$\overrightarrow{PQ} = \vec{R} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

Here, $(x_2 - x_1), (y_2 - y_1), and\ (z_2 - z_1)$ are called as components of the vector

$$|\vec{R}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

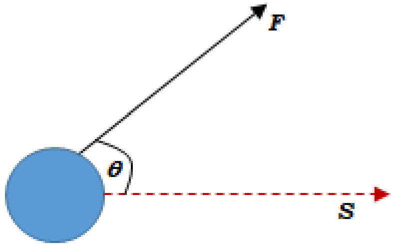
$$\hat{a} = \frac{\vec{R}}{|\vec{R}|} = \frac{(x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

Vector multiplication

A vector can be multiplied by another vector in two different ways namely (i) Scalar product (dot product) (ii) Vector product (cross product)

I. Scalar Product (dot product):

Scalar product or dot product of two vectors is defined as the product of their magnitudes and cosine of the angle between them. Scalar product of two vectors results in a number (a scalar)



If $\vec{P}$ and $\vec{Q}$ are two non-zero vectors acting at point at an angle θ then

$$\vec{P}.\vec{Q} = |\vec{P}||\vec{Q}|cos\theta$$

Work done (W) by the applied force on a rigid body is an example for scalar product and is given by the scalar product of force (F) and displacement (S)

$$W = \vec{F}.\vec{S} = |\vec{F}||\vec{S}|cos\theta$$

Electric flux, another example, is the scalar product of electric intensity and surface area. In scalar product we have

$$\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0, \hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$$

Basic properties of scalar product are

i. $\vec{P}.\vec{Q} = \vec{Q}.\vec{P}$	Commutative property
ii. $(k\vec{P}).\vec{Q} = \vec{P}.(k\vec{Q}) = k(\vec{P}.\vec{Q})$	Associative property
iii. $\vec{P}.(\vec{Q} + \vec{R}) = \vec{P}.\vec{Q}+\vec{P}.\vec{R}$	Distributive property
iv. $\vec{P}.0 = 0.\vec{P} = 0$	

II. Vector product (Cross product):

Vector product or cross product of two vectors is defined as the product of their magnitudes and sine of the angle between them. The vector product of two vectors always gives a vector quantity. If $\vec{P}$ and $\vec{Q}$ are two non-zero vectors acting at point at an angle θ then

$$\vec{P}\times\vec{Q} = |\vec{P}||\vec{Q}|\sin\theta$$

The cross product of $\vec{P}$ and $\vec{Q}$ is always perpendicular to the plane containing $\vec{P}$ and $\vec{Q}$

Torque (τ) acting on a dipole placed in the electric field is given by the cross product of field intensity (E) and dipole moment (p)

$$\vec{\tau} = \vec{p}\times\vec{E} = |\vec{p}||\vec{E}|\sin\theta$$

Force acting on a charged particle moving in the external magnetic field is given by the cross product of velocity and magnetic field intensity. In vector product we have

$$\hat{\imath}\times\hat{\jmath} = \hat{k}, \quad \hat{\jmath}\times\hat{k} = \hat{\imath}, \quad \hat{k}\times\hat{\imath} = \hat{\jmath}, \quad \hat{\imath}\times i = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0$$

In cross product the direction of the resultant vector is given by thumb rule

Basic properties of vector product are

i.	$\vec{P} \times \vec{Q} = -\overrightarrow{(Q \times P)}$	Anti-commutative property
ii.	$(k\vec{P}) \times \vec{Q} = \vec{P} \times (k\vec{Q}) = k(\vec{P} \times \vec{Q})$	Associative property with respect to scalar
iii.	$\vec{P} \times (\vec{Q} \times \vec{R}) \neq (\vec{P} \times \vec{Q}) \times \vec{R}$	Not Associative property with respect to vector
iv.	$\vec{P} \times (\vec{Q} + \vec{R}) = \vec{P} \times \vec{Q} + \vec{P} \times \vec{R}$	Distributive property
v.	$\vec{P} \times 0 = 0 \times \vec{P} = 0$	

Scalar and Vector fields

A point in space is described by three independent parameters. A continuous function of the position of a point in space is known as *point function.* The region of space in which the point function specifies a physical quantity is known as *field*.

Any field, for that matter, always represents region of influence. Consider some examples. Electrostatic force around a charged body, temperature of air around the flame, density of gas that makes up the body of a star, flow of incompressible fluid, pressure of air on the earth etc. In these examples, each point function represents a physical quantity. The point function may have only magnitude or both magnitude and direction. Based on the nature of the point function, *field* is classified as **Scalar field** and **Vector field**

Scalar field:

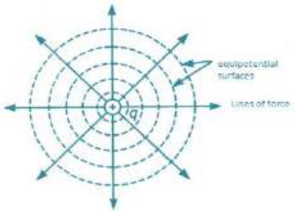
Scalar field is a region of space where each point is associated with scalar point function i.e., it has only magnitude and does not depend on how the coordinates are chosen. It is represented as f(x).

In the above mentioned examples density of gas, temperature of air and pressure of air at a given point are clearly function of position and does not consider coordinates. Hence they are scalar fields. Scalar fields are represented by drawing surfaces.

Ex: density of gas, temperature and pressure of air, Electrostatic potential

In the above mentioned examples density of gas, temperature of air and pressure of air at a given point are clearly function of position and does not consider coordinates. Hence they are scalar fields. Scalar fields are represented by drawing surfaces.

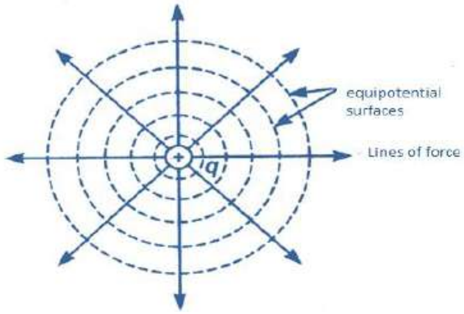
Electrostatic potential around a charged body is also a scalar field. Each surface that passes through all the points having same scalar quantity is known as *equal* or *level* surface. Equipotential surface is one such surface. It passes through all the points at same electric potential around a charged body



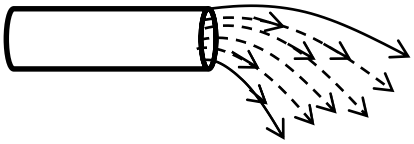
Vector field:

Vector field is a region of space where each point is associated with vector point function i,e, it has magnitude and direction; both of which change from point to point. It is represented as *v(x)* At any instant, velocity vector of water flow, velocity vector of a rotating body, force acting on a test charge in the electric field, intensity of magnetic field around a bar magnet, etc are the examples for vector field.

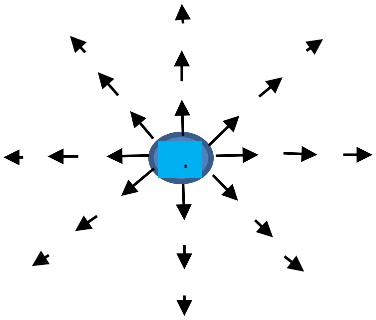
Vector field is represented by *vector lines, lines of surface*. The tangent at a vector line gives the direction of the vector at that point



Scalar field



Vector field



Vector field

There are two mathematically important properties of vector field namely ***circulation*** and ***flux.*** They are frequently used to describe the laws of electricity and magnetism.

Circulation: For any vector field the *circulation* around any imaginary closed curve is defined as the product of average tangential component of the vector and circumference of the loop

$$circulation = \vec{v}_1.\overrightarrow{dl}_1 + \vec{v}_2.\overrightarrow{dl}_2 + \vec{v}_3.\overrightarrow{dl}_3 + ---= \oint \vec{v}_i.\overrightarrow{dl}_i$$

Flux: For an arbitrary closed surface, the *flux* (either outward or inward) is the product of average normal component of the vector and surface area. The outward flux is positive and inward flux is negative.

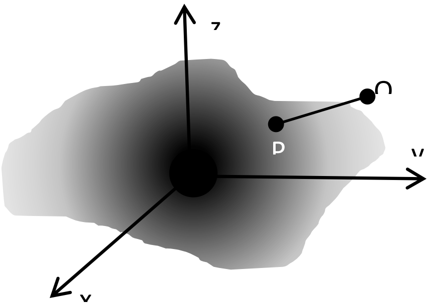
$$flux = \vec{v}_1.\overrightarrow{dS}_1 + \vec{v}_2.\overrightarrow{dS}_2 + \vec{v}_3.\overrightarrow{dS}_3 + ---= \oint \vec{v}_i.\overrightarrow{dS}_i$$

Vector Calculus

Vector calculus includes both vector functions (vector fields) and scalar functions (Scalar fields). Vector calculus owes much of its importance in engineering and physics to the **gradient**, **divergence** and **curl**. Some of the vector fields (not all) can be obtained from scalar fields by performing specific operations on them.

When field vary with respect to time, we describe the variation by taking their derivatives with respect to time. Similarly, if a point function in the field varies with respect to position, we can describe the variations with respect to position. This is nothing but *directional differentiation.*

Consider the case of variation of temperature at different positions around a red hot iron sphere. It is decreasing along x, y and z axes. The rate at which it decreases along x – axis does not depend on y & z axes. Similarly the decrease rate along remaining 2 axes also independent of other axes.



Let *T(x,y,z)* be the temperature (scalar) a*t* a point in space. Then $\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}$ & $\frac{\partial T}{\partial z}$ denote the rate of change of T along x,y & z axes

Now consider two points P and Q in space at which the temperature is $T_1(x,y,z)$ and $T_2(x+dx, y+dy, z+dz)$ then the variation of temperature from P to Q is given by

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz - - - - (1)$$

Here we introduce an operator called '*del operator' (∇)* as

$$\nabla = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)$$

and eq (1) is rewritten as

$$gardT = \nabla T = \frac{\partial T}{\partial x}i + \frac{\partial T}{\partial y}j + \frac{\partial T}{\partial z}k$$

DEL operator

Like any other operators in mathematics like addition, subtraction, differentiation, summation, integration etc, ∇ (del) is also an operator. It is a vector operator which operates on both scalars and vectors. It is written as

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)$$

∇ alone, it means nothing. When multiplied by a scalar, then also it is nothing. If a scalar is multiplied by ∇ then it carries a meaning. In ordinary algebra

TA=AT

Where T is scalar and A is vector. But the same is not true in the case of ∇ operator

i,e $T\nabla \neq \nabla T$

∇ obeys the same convention as the derivative notation. What is to be differentiated must be placed on the right side of the∇**.** When operated on scalar and vector we obtain three parameters namely gradient, divergence and curl.

Note:

- ∇ Operator is a vector. When a vector is multiplied by a scalar, the resultant is vector. Multiplying vector by another vector via dot product yields a scalar and multiplying a vector by a vector via cross product gives a vector quantity.
- Similarly ∇ gives a vector quantity (∇T-grad) if multiplied with a scalar T; scalar ($\nabla.\vec{h}$-Div) if operated on a vector $\vec{h}$ by dot product; and a vector ($\nabla \times \vec{h}$- curl) if operated by cross product

Gradient of scalar field

Gradient at any point in the scalar field is equal to the rate of change of scalar along the normal to the surface at that point. If *f(x,y,z)* is scalar point function then

$$grad\ f = \nabla f$$

$$grad\ f = \nabla f = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) f$$

Explanation:

Consider the case of temperature variation around a red hot iron block (as mentioned above). The decrease in the temperature along any direction represents the gradient (directional differentiation). As we move away from the block, temperature keeps decreasing *gradually*. This is nothing but negative gradient. On the other hand, if we move towards the block, temperature increases and hence the gradient is positive.

Note: Equation (1) can be written as

$$dT = \left(\frac{\partial T}{\partial x}i + \frac{\partial T}{\partial y}j + \frac{\partial T}{\partial z}k\right).(dxi + dyj + dzk)$$

$$dT = \nabla T.dr = |\nabla T||dr|cos\theta$$

Here, ∇T is gradient, *dr* is position vector and θ is the angle between them. According to this equation

i. If θ=0, then ∇Tis positive and dT is maximum, ie, gradient points in the direction of maximum increase of scalar function T
ii. If θ = 180, then ∇Tis negative. T decreases in the direction of ∇T

Physical significance

i. Gradient of a scalar field is a vector. It is directed along the increasing scalar field
ii. In the case of red hot iron sphere ∇T increases as we move towards the sphere. Hence *T(x,y,z)* also increases
iii. If *T(x,y,z)* is a continuously differentiable real value function like temperature around the red hot iron sphere, then $\nabla T \neq 0$
iv. Electric field is actually negative gradient of potential
v. There is no meaning for gradient of a vector

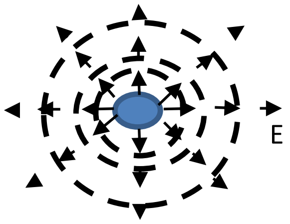
There exist electrostatic potential (V) and electric field intensity (E) around a charged body. From the fundamentals of electrostatics, we have learnt that potential and electric intensity are related as (along x-axis)

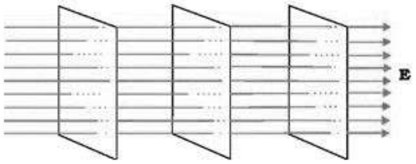
$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{a}_x\right)$$

In 3-d Cartesian system

$$\vec{E} = -\left(\frac{\partial V}{\partial x}i + \frac{\partial V}{\partial y}j + \frac{\partial V}{\partial z}k\right) = -\nabla V$$

Electric intensity is also called as "negative gradient of electric potential". The negative sign indicates that the potential decreases in the direction of electric field intensity. If the body is positively charged electric intensity is radially outward and as we move away from charged body potential decreases as shown in the following figure.





Divergence of a vector field

From the discussion of gradient operation we came to know that when vector operator ∇ operates on a scalar, it yields a vector quantity called *gradient*. Gradient simply represents the maximum rate of increase of scalar field and directed normal to the surface. Now the question is, can we do any algebra with the vector∇?. If the answer is yes, what will be the resultant quantity?. Here is the explanation.

One can take the dot product of ∇ and a vector *v* (or ***h***). The resultant is termed as *divergence* and it is always a scalar
Divergence of a vector point function is the dot product of ∇ and a vector. If h(*x*,*y*,*z*) is a vector point function in 3-D space, then its divergence is given by

$$divV = \nabla.h = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(h_x\hat{\imath} + h_y\hat{\jmath} + h_z\hat{k}\right)$$

It is also defined as the surface integral of the flux per unit volume as dV tends to zero. Hence divergence of a vector is also written as

$$\vec{\nabla}.\vec{h} = \lim_{dV\to 0}\frac{\oint \vec{h}.\overrightarrow{dS}}{dV}$$

Explanation:

Consider a vector field, say flow of water. Imagine a closed surface in this field and assume that you are sitting inside the surface. Now ask a question; whether I am losing 'something' from inside? (or, does the outflow is more than inflow?). It can be decided by looking at the velocity vectors normal to the surface. If there are more outward velocity vectors than inward then, yes, you are losing something. In this context we can define a quantity called *flux* which is intimately related to *divergence*.

The amount of outward (or inward) flow of fluid through the surface per unit time is known as "flux". It is calculated by multiplying the average normal component and the surface area (dot product).

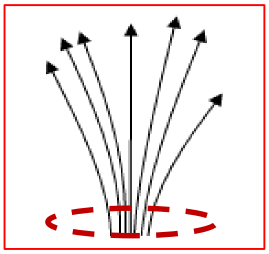
$$flux = (\text{normal component}).(\text{surface area})$$

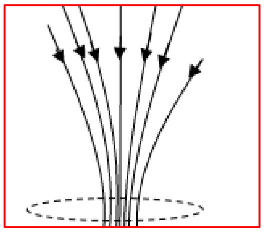
In the above situation, consider a small surface area which encloses volume dV. The net out flow over this surface is obtained by integrating the flux over the surface (surface integral). The surface integral of the flux per unit volume as dV tends to zero is known as *divergence*. Hence divergence of a vector is also written as

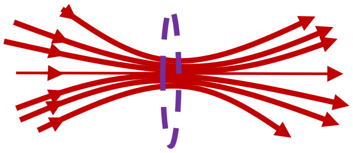
$$\vec{\nabla}.\vec{h} = \lim_{dV\to 0} \frac{\oint \vec{h}.\overrightarrow{dS}}{dV}$$

If $\vec{E}$ is the vector field (electric field), $\overrightarrow{dS}$ is the area vector and V is the volume enclosed by the surface then

$$\vec{\nabla}.\vec{E} = \frac{\oint \vec{E}.\overrightarrow{dS}}{V}$$







Physical significance

i. Divergence of a vector represents 'how much of the vector filed is flowing out' (or flowing in) over a surface area. It is always associated with flux.

ii. If the net flow is outward then divergence is positive. If the net flow is inward then the divergence is negative

iii. The point of positive divergence is "source" and the point of negative divergence is "sink"

iv. If the flow is steady (net inflow is equal to out flow) then divergence is ZERO, $\nabla.h = 0$. In such case the vector function is termed as ***incompressible*** or ***solenoidal***

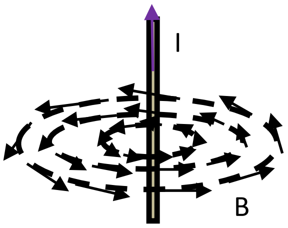
v. There is no meaning for divergence of a scalar

Curl of a vector field

As mentioned earlier (in vector field section), *circulation* is one of the two mathematically important properties of vector field. In the case of magnetic field around a straight conductor carrying current or whirlpool in which water mass is rotating, vector function is rotating about a fixed axis. The term *circulation* is associated with such kind of rotating vector fields and the rotational effect is measured in terms of "curl

Curl of a vector point function is the cross product of ∇ and a vector. If $\vec{B}(x, y, z)$is a vector point function in 3-D space, then its curl is given by

$$\text{curl } \vec{B} = \nabla \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$



Curl of a vector is also a vector whose direction is obtained by right hand rule(perpendicular to the plane containing the two vectors namely ∇ and $\vec{B}$)

Curl is the measure of net circulation and in this context we can write curl as

$$curl\vec{B} = \nabla \times \vec{B} = \lim_{S \to 0} \left(\frac{\oint \vec{B}.dl}{\vec{S}} \right)$$

It is explained as follows

Consider the case of magnetic field produced by a straight conductor carrying current. Imagine a loop in this field and draw tangents at different points on the loop. When you observe these tangents, you feel they are rotating (circulating) about a particular axis. Take the integral of these components all the way around the loop. The integration gives *circulation* of the vector field around the loop.

$$circulation = \oint \vec{B}.dl$$

The measure of net circulation is known as **curl.** *Circulation* goes on decreasing as we reduce the size of the loop and area enclosed by the loop(s). Finally it reaches a limiting value when the area approaches to zero. In this context we can write curl as

$$curl\vec{B} = \nabla \times \vec{B} = \lim_{S \to 0} \left(\frac{\oint \vec{B}.dl}{\vec{S}} \right)$$

Physical Significance

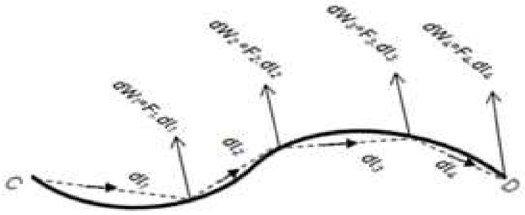
i. Curl of a vector represents the rotational effect of the vector field.
ii. If the field is uniform (like electric field due to infinite charge) the curl is zero. A vector field is said to be ***irrotational*** if the curl is zero ie, $\nabla \times \vec{B} = 0$
iii. For certain vector fields, (non uniform fields) the curl of a vector is non-zero, meaning thereby that they have the capability to rotate.
iv. Curl is a measure of the vorticity of the field at the point (a measure of the rate of rotational spin in a fluid)
v. There is no meaning for curl of a scalar

Types of Integrations

Integration is a process of bringing things together which were lying between two limits. These things may be spread over a line or surface or in space. Accordingly we have ***line integral, surface integral & volume integral***

Line integral

A **Line integral** is an integration of a function along a curve. One can integrate scalar valued functions (like mass of a wire) or vector valued functions (like work done by a force) along a curve In the following fig, work done by a force F in displacing an object along the path CD is calculated using line integration as follows

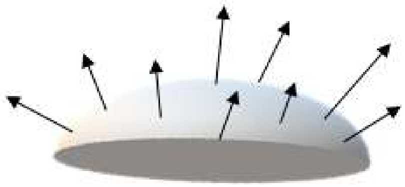


$$W = F_1 . dl_1 + F_2 . dl_2 + F_3 . dl_3 + \; - - = \int_C^D F . dl$$

Surface integral

In **Surface integral**, we add up the vector components that are flowing through surface area. Like line integral stands for work done, surface integral represents **flux.** Hence it is also known as **flux integral**

In the following fig, flux is calculated over a closed surface by dividing it into small surfaces each of area dS_i



$$flux\ (\Phi) = \vec{v}_1 . \overrightarrow{dS}_1 + \vec{v}_2 . \overrightarrow{dS}_2 + \vec{v}_3 . \overrightarrow{dS}_3 + \; - - - = \oint \vec{v}_i . \overrightarrow{dS}_i$$

Volume integral

A **volume integral** refers to an integral over a 3-D domain, that is, it is a special case of multiple integrals (Wikipedia definition). It is used to find the volume of solids, volume of revolution, etc

$$V = \oint A dV$$

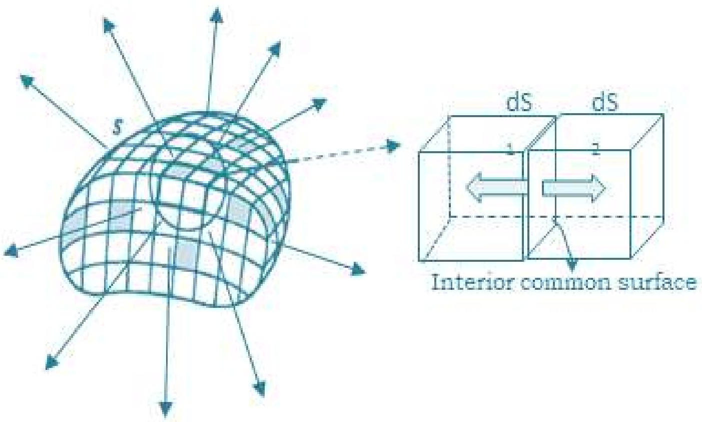
Gauss divergence theorem:

This theorem was originally started from the study of fluid and It helps to convert volume integral into surface integral

Statement:

The surface integral of flux of a vector field $\vec{E}$ (say electric field) over any closed surface S is equal to volume integral of the divergence of the same field. The mathematical form of divergence theorem is as follows

$$\oint \vec{\boldsymbol{E}} . \overrightarrow{\boldsymbol{dS}} = \oint (\boldsymbol{\nabla} . \vec{\boldsymbol{E}}) \boldsymbol{dV}$$



Proof:

Consider a vector field $\vec{E}$ in space. Take many adjoining incremental volumes of any shape, we form a macroscopic volume V with enclosing surface $\boldsymbol{S}$ as shown in Fig. However, each interior common surface between incremental volumes has the flux leaving one volume (positive flux contribution) just entering the adjacent volume (negative flux contribution) as in Fig. The net contribution to the flux is zero for all interior surfaces. Nonzero contributions to the flux are obtained only for those surfaces which bound the outer surface **S** *of V* (shaded area)

The total flux is obtained by adding all contributions from each differential volume. Let dS_1, dS_2, dS_3, etc are the surface areas of the faces of small incremental volumes. The net flux is given by

$$\oint \vec{E}.\overrightarrow{dS} = \oint \vec{E}.\overrightarrow{dS}_1 + \oint \vec{E}.\overrightarrow{dS}_2 + \oint \vec{E}.\overrightarrow{dS}_3 + --$$

$$\oint \vec{E}.\overrightarrow{dS} = \sum_{i=1}^{N} \oint \vec{E}.\overrightarrow{dS}_i$$

$$\oint \vec{E}.\overrightarrow{dS} = \sum_{i=1}^{N} V \frac{\oint \vec{E}.\overrightarrow{dS}_i}{V} ----(1)$$

By definition of divergence,

$$\vec{\nabla}.\vec{E} = \frac{\oint \vec{E}.\overrightarrow{dS}}{V} ----(2)$$

From (1) and (2) we get

$$\oint \vec{E}.\overrightarrow{dS} = \sum_{i=1}^{N} V(\vec{\nabla}.\vec{E}) ----(3)$$

In limiting case we can write

$$\sum_{i=1}^{N} V = \oint dV$$

Hence equation (3) becomes

$$\oint \boldsymbol{\vec{E}}.\boldsymbol{\overrightarrow{dS}} = \oint (\boldsymbol{\vec{\nabla}}.\boldsymbol{\vec{E}})\boldsymbol{dV}$$

Stokes' Theorem

This theorem helps us to convert line integral into surface integral

Statement:

The line integral (or circulation) of a vector field **F** around a fixed macroscopic path is equal to the surface integral of curl **F** over a surface bounded by the path.

$$\oint \vec{\boldsymbol{F}}.\overrightarrow{\boldsymbol{dl}} = \oint (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}})\overrightarrow{\boldsymbol{dS}}$$

This theorem helps us to convert line integral into surface integral

Maxwell's Equations

Introduction:

Two branches of physics namely electricity and magnetism developed quite separately until Oerstead showed that a current carrying conductor can produce magnetic field around it(1820). With this discovery, a new branch called "*electromagnetism"* came into existence. It is indeed an amalgamation of three phenomena namely *electrostatics, current electricity & magnetism.* Oerstead, Ampère, Henry, Faraday and others have contributed to this area. The basic phenomena and the connection between the three disciplines were ultimately described by Maxwell (1831–1879) in four famous equations popularly known as ***Maxwell's equations***.

It is now time to discuss *'**God's own equations'*** (according to Boltzmann) and ***'most significant event of 19th century'*** (according to Richard Feynman)- Maxwell's equations. They are as important in electrodynamics as Newton's laws of motion in classical mechanics

Gauss's theorem in electrostatics (Maxwell's equation-I)

Maxwell's first equation is derived immediately from Gauss's theorem in electrostatics as follows

The surface integral of net flux is equal to (1/ε₀) times the charge enclosed by the surface

$$\oint \vec{E}.\overrightarrow{dS} = \frac{1}{\varepsilon_0}(q_{encl})$$

Using divergence theorem we can reduce this equation to

$$\therefore \vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{D}} = \boldsymbol{\rho}$$

This is known as Maxwell's 1st equation

Here ρ is known as volume charge density and $\vec{D} = \varepsilon_0 \vec{E}$ called displacement vector

Gauss theorem in magnetism (Maxwell equation –II)

The Gauss theorem in magneto statics gives us the second Maxwell equation

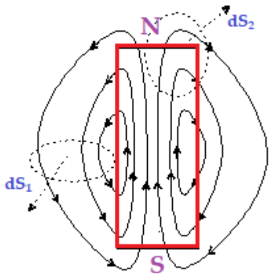
The net flux of the magnetic field through a closed surface is zero irrespective of whether it encloses any source or not

$$\oint \vec{B}.\overrightarrow{dS} = 0$$

Again using divergence theorem we can reduce this equation to

$$\therefore \vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{B}} = \boldsymbol{0}$$

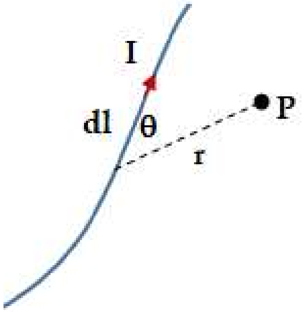
This is known as Maxwell's 2nd equation



Biot-Sovart law.

Magnetic field produced by a current element is given by Biot-Sovart law. It states that
The magnitude of field produced at a point due to current carrying conductor is directly proportional to current in the conductor, length of the conductor, sine of the angle between line joining the point and direction of the current and inversely proportional to square of the distance

$$dB = \left(\frac{\mu_0}{4\pi}\right)\frac{Idl\sin\theta}{r^2}$$



Biot Savart law is similar to Coulomb's law. Both show linear relationship between source and field. Both are inverse square laws. The main difference is that Coulomb's law does not speak about the direction of the field.

Current element:

The product of magnitude of current and small length of the conductor (Also called as differential length) through which current flows is known as *current element* (*Idl*). As charge is for electric field, current element is for magnetic field.

Law of Electromagnetic induction (Maxwell equation –III)

Maxwell's 3rd equation is a direct consequence of laws of Electromagnetic induction. Law of EMI states that
An emf is induced in a coil due to continuous change in the magnetic flux linked with the coil and magnitude of induced emf (e) is directly proportional to rate of change of flux (Φ)

$$e = -\frac{d\Phi}{dt}$$

From this equation we can show that the induced electric field **E** is

$$(\nabla \times \vec{E}) = -\frac{\partial \vec{B}}{\partial t}$$

This is Maxwell's 3rd equation

Continuity:

According to Maxwell *"A quantity is said to be a continuous function of its variables when, if the variables alter continuously, the quantity itself alters continuously".*
We shall now find the continuity of current in terms of current density

Continuity of Current:

Statement: The total current flowing out of some volume is equal to the rate of decrease of charge within that volume. The mathematical form of this statement is

$$\nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t}$$

This is the equation of continuity.

LHS represents the divergence of *J*. If the divergence of **J** is positive, then more charge is leaving than that entering the specified volume. If charge is leaving, then the amount of charge within the volume must be decreasing. This is exactly what the RHS is.

Ampere's Circuital theorem

Ampere's circuital law is very similar to the Gauss' Law in electrostatics.

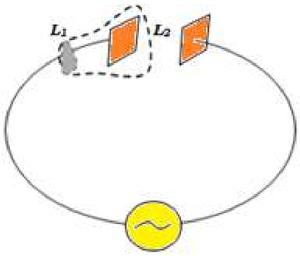
The line integral of tangential component of the magnetic field strength around a closed path is equal to μ_0 *times the current enclosed by the path.*

$$\oint \vec{B}.\vec{dl} = \mu_0 I ----(1)$$

It can also be written as

$$\oint \vec{H}.\vec{dl} = I ----(1)$$

This theorem fails in the case of varying fields. Look at the case of parallel plate capacitor connected to AC source. Imagine a pot shaped loop around one of the plate. When Ampere's law is applied over loop L_2 the RHS of eq (1) becomes zero. But actually there is magnetic field in the gap between the plates. In order to explain this contradiction Maxwell introduced ***Displacement current***



Displacement current (I_D)

While correcting the Ampere's law for time varying fields, Maxwell introduced a new quantity called displacement current density (J_D). This is 3rd kind of current density. [The other two are conduction current density ($J=\sigma E$) and convection current density ($J=\rho_v V$)].It leads to a new type of current called **displacement current** (I_D).

Displacement current is defined as the current produced by time varying electric fields

It is same as conduction current (I) arising due to flow of charges in conducting wire and capable of producing all the effects. The expression for I_D is obtained as follows

The electric field between two plates of a parallel plate capacitor is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{q}{A\varepsilon_0}$$

Flux due to this field is

$$\Phi = EA = \frac{qA}{A\varepsilon_0} = \frac{q}{\varepsilon_0}$$

$$\frac{d\Phi}{dt} = \frac{1}{\varepsilon_0}\frac{dq}{dt}$$

$$\frac{dq}{dt} = \varepsilon_0 \frac{d\Phi}{dt}$$

LHS of the above equation is nothing but electric current. Hence RHS also shall represent current and it is known as ***displacement current (I_D)***

$$I_D = \varepsilon_0 \frac{d\Phi}{dt}$$

This is the expression for displacement current

Ampere-Maxwell law (Maxwell equation –IV)

The modified Ampere's law, now called as Ampere-Maxwell law, is therefore

$$\oint \boldsymbol{H.dl} = \boldsymbol{I} + \boldsymbol{I_D}$$

Here I = conduction current; I_D = displacement current

Using Stokes' theorem we can reduce this equation to

$$\overrightarrow{\nabla} \times \vec{H} = \vec{J}$$

This is Maxwell's 4th equation

Difference between I_D and I

i. Displacement current is due to time varying fields where as conduction current is due to steady fields

ii. Displacement current does not require any material medium for the flow whereas conduction current requires conducting wires for the flow

iii. Displacement current can exists even in free space (Vacuum) whereas conduction current do not have this privilege

iv. Displacement current is basically rate of change of electric flux whereas conduction current is the rate of flow of charge

Maxwell's equations in statement form (only for steady field)

Maxwell's equation-I

Divergence of electric flux density over a closed surface is equal to the volume charge density enclosed by the surface.

$$\nabla.\vec{D} = \rho$$

Maxwell's equation-II

Divergence of magnetic flux density over a closed surface is always zero

$$\nabla.\vec{B} = 0$$

Maxwell's equation-III

Curl of the induced electric filed is equal to negative rate of change of magnetic flux density

$$(\nabla \times \vec{E}) = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell's equation-IV

The curl of magnetic field induced is equal to current density.

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

Steady Field		
Eq.No	**Point form**	**Source**
1	$\nabla . \vec{D} = \rho$	Gauss theorem in electrostatics
2	$\nabla . \vec{B} = 0$	Gauss theorem in Magnetism (Non-existence of monopoles)
3	$(\nabla \times \vec{E}) = 0$	Laws of EMI
4	$\overrightarrow{\nabla} \times \vec{H} = \vec{J}$	Ampere's circuital law

Plane EM waves

Introduction:

Two most significant and remarkable outcomes of Maxwell's equations are (i) existence of displacement current (ii) prediction of electromagnetic wave (EM wave).

When **E** and **H** vary in two mutually perpendicular planes then the net energy propagates in the form of a wave in a direction perpendicular to both **E** & **H**. This wave is known as ***uniform plane wave***

A ***uniform plane wave*** is one

i. Which contain both electric field (**E**) and magnetic field (**H**) in transverse planes.
ii. The normal to the transverse plane represents the direction of propagation.
iii. The variation of **E** & **H** is only along the direction of propagation – ie, normal to the transverse plane and remain same in their respective planes.

We shall now obtain an equation for EM waves using Maxwell's equations.

Expression for EM wave

Consider Maxwell's equation in homogeneous medium

$$(\nabla \times \vec{E}) = -\frac{\partial \vec{B}}{\partial t} = -\frac{\mu \partial \vec{H}}{\partial t}$$

$$\nabla \times \nabla \times \vec{E} = -\frac{\mu \partial (\nabla \times \vec{H})}{\partial t}$$

But $\nabla \times (\nabla \times h) = \nabla(\nabla . h) - \nabla^2 h$

$$\therefore \nabla(\nabla . \vec{E}) - \nabla^2 \vec{E} = -\frac{\mu \partial (\nabla \times \vec{H})}{\partial t}$$

$$\text{But } \nabla . \vec{D} = \rho_v \Rightarrow \nabla . \vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\therefore \nabla \left(\frac{\rho_v}{\varepsilon}\right) - \nabla^2 \vec{E} = -\frac{\mu \partial (\nabla \times \vec{H})}{\partial t}$$

Put $\nabla \times \vec{H} = \vec{J} + \frac{\varepsilon \partial \vec{E}}{\partial t}$

$$\nabla\left(\frac{\rho_v}{\varepsilon}\right) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left(\vec{J} + \frac{\varepsilon\partial\vec{E}}{\partial t}\right) = -\mu\frac{\partial\vec{J}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = \mu\frac{\partial\vec{J}}{\partial t} + \nabla\left(\frac{\rho_v}{\varepsilon}\right) - - - - (1)$$

In free space, J=0 and ρ=0. Hence

$$\nabla^2\vec{E} - \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = 0 - - - - (2)$$

Equation (2) is in the form wave equation travelling with velocity *v*

$$\nabla^2 f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 - - - - (3)$$

Hence we conclude that equation (2) represents the uniform plane electromagnetic wave travelling in free space. Its velocity is given by

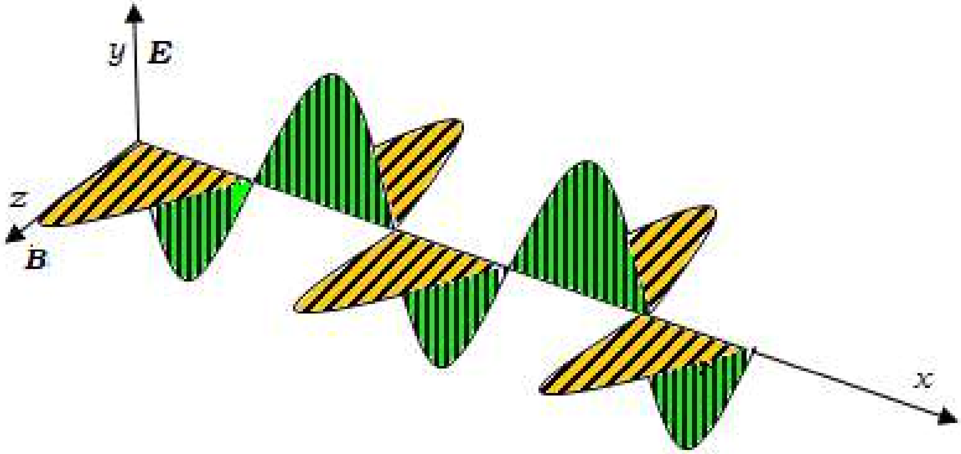
$$v = c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} - - - - (4)$$

An EM wave propagates along *x*- axis in homogeneous medium along with **E** parallel to *y*- axis and **B** along *z*- axis. Both **E** and **B** shall vary along *x*-axis. Their instantaneous values are given by

$$E(x,t) = E_0\sin(kx - \omega t)\hat{a}_y$$

$$B(x,t) = B_0\sin(kx - \omega t)\hat{a}_z$$

Here *K = wave number; ω = angular frequency*



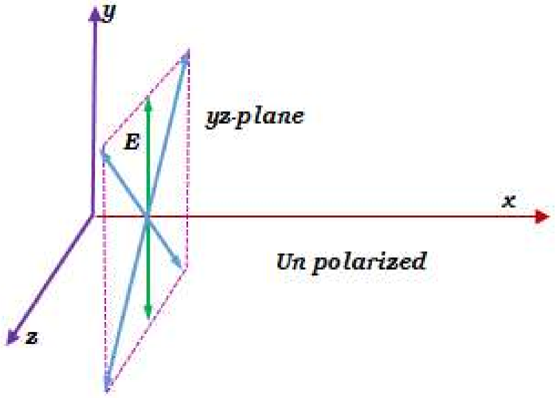
Properties

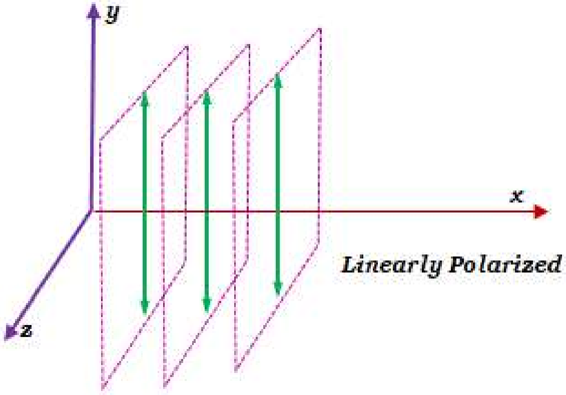
- EM waves consists of vibrating electric and magnetic fields. Hence they are transverse
- They travel with a speed of 3×10^8m/s in free space or vacuum
- They transmit energy through matter or across space.
- They undergo reflection, refraction, diffraction, polarization
- Photoelectric effect, Compton effect and pair production are some exclusive properties of EM waves
- Although all electromagnetic waves travel at the same speed across space, they may differ in their wavelengths, frequencies, and energy levels
- They exert pressure on the surface of incidence. It is called *radiation pressure*

Polarization

The electric component of an electromagnetic plane wave can oscillate in any direction perpendicular to the direction of wave propagation. Suppose that the wave is propagating in the *x*-direction, the electric field can oscillate in any direction that lies in the *y*-*z*-plane. Such wave is said to be ***Un polarized***

If the electric field is restricted to any one direction then the wave is said to be ***polarized***





There are three types of polarization namely

- ***linear polarization,***
- ***circular polarization***
- ***elliptical polarization***

i. When the orientation of the **E**-field vector does not change, we say that the wave is ***linearly polarized***. Its tip along a particular axis describes a straight line. Hence it is called *linear polarization*

ii. If we combine two linearly polarized waves of equal amplitude of **E** vector having a phase difference of $\pi/2$; one polarized in the y -direction, and one in the z-direction, then we obtain a ***circularly polarized*** wave. The tip of the resultant **E** vector traces out a circle in the plane perpendicular to direction of propagation. Hence the name is circular polarization

iii. If we combine two linearly polarized waves of unequal amplitude of **E** vector having a phase difference of $\pi/2$; one polarized in the y -direction, and one in the z-direction, then we obtain a ***elliptically polarized*** wave. The tip of the resultant **E** vector traces out an ellipse in the plane perpendicular to direction of propagation. Hence the name is circular polarization

Solved Numericals

1. Find the gradient of *f(x,y)=xy*^{2}*+x*^{3}*y* at the points (1,2)

Solution:

$$\nabla f = \left(\frac{\partial(xy^2+x^3y)}{\partial x}i + \frac{\partial(xy^2+x^3y)}{\partial y}j\right)$$

$$\nabla f = (y^2+3x^2y)\hat{a}_x + (2xy+x^3)\hat{a}_y$$

Put x =1, y=2,

$$\nabla f = (2^2+3(2))i + (1^3+4)j = 10i+5j$$

$$\nabla f = 10i+5j = (10{,}5)$$

2. The force in an electric field is *f(x,y)*= *x*^{2} - *y*^{2}. Find the gradient of *f* at P(-1,3)

Solution:

$$\nabla f = \left(\frac{\partial(x^2-y^2)}{\partial x}i + \frac{\partial(x^2-y^2)}{\partial y}j\right)$$

$$\nabla f = (2xi\ -2yj)$$

$$\nabla f = (2(-1)i\ -2(3)j) = (-2,-6)$$

3. Let a vector field be $\boldsymbol{\vec{P} = x^2i + y^2j + z^2k}$. Find the divergence

Solution:

$$\nabla.\vec{P} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).(x^2i+y^2j+z^2k)$$

$$\nabla.\vec{P} = \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z}$$

$$\nabla.\vec{P} = 2x+2y+2z = 2(x+y+z)$$

4. Let the velocity vector is $\boldsymbol{\vec{v} = y\hat{a}_x}$. Show that the flow is incompressible

Solution:

The flow is said to be incompressible only if $\nabla.\vec{v} = 0$

$$\nabla.\vec{v} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).(yi)$$

$$\nabla.\vec{v} = \frac{\partial y}{\partial x} = 0$$

Hence the flow is incompressible

5. Let the vector point function is $\boldsymbol{\vec{P} = yzi + zxj + xyk}$ Show that it is solenoidal

Solution:

Condition for solenoidal $\nabla.\vec{P} = 0$

$$\nabla.\vec{P} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).(yzi+zxj+xyk)$$

$$\nabla.\vec{P} = \left(\frac{\partial(yz)}{\partial x} + \frac{\partial(zx)}{\partial y} + \frac{\partial(xy)}{\partial z}\right) = 0$$

Hence the given vector is solenoidal

6. Let the vector point function is $\vec{P} = yzi + zxj + xyk$. Find the curl. Is it rotational or irrotational?

Solution:

$$\text{curl } \vec{P} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$curl\ \vec{P} = i\left(\frac{\partial(xy)}{\partial y} - \frac{\partial(zx)}{\partial z}\right) - j\left(\frac{\partial(xy)}{\partial x} - \frac{\partial(yz)}{\partial z}\right) + k\left(\frac{\partial(zx)}{\partial x} - \frac{\partial(yz)}{\partial y}\right)$$

$$curl\ \vec{P} = i(x-x) - j(y-y) + k(z-z) = 0$$

Curl is zero. Hence the given vector field is irrotational

7. Show that the electric field represented by a vector $\vec{E} = (z^2 + 2x + 3y)i + (3x + 2y + z)j + (y + 2zx)k$ is irrotational

Solution:

$$\text{curl } \vec{P} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (z^2 + 2x + 3y) & (3x + 2y + z) & (y + 2zx) \end{vmatrix}$$

$$\text{curl } \vec{P} = i\left(\frac{\partial(y+2zx)}{\partial y} - \frac{\partial(3x+2y+z)}{\partial z}\right) - j\left(\frac{\partial(y+2zx)}{\partial x} - \frac{\partial(z^2+2x+3y)}{\partial z}\right)$$
$$+ k\left(\frac{\partial(3x+2y+z)}{\partial x} - \frac{\partial(z^2+2x+3y)}{\partial y}\right)$$

$\text{curl } \vec{P} = i(1-1) - j(2z-2z) + k(3-3) = 0$

Hence the given field is irrotational

8. The flux density is given as $\vec{D} = \frac{1}{z^2}\left(10xyz\hat{a}_x + 5x^2z\hat{a}_y + (2z^3 - 5x^2y)\hat{a}_z\right)$ in cartesian coordinates. Find the divergence at P(-2,3,5)

Solution:

In Cartesian coordinate system divergence is given by

$$\nabla.\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_Z}{\partial z}$$

$$\nabla.\vec{D} = \frac{1}{z^2}\left(\frac{\partial(10xyz)}{\partial x} + \frac{\partial(5x^2z)}{\partial y} + \frac{\partial(2z^3 - 5x^2y)}{\partial z}\right)$$

$$\nabla.\vec{D} = \frac{yz}{z^2}\frac{\partial(10x)}{\partial x} + \frac{x^2z}{z^2}\frac{\partial(5)}{\partial y} + \frac{1}{z^2}\frac{\partial(2z^3 - 5x^2y)}{\partial z}$$

$$\nabla.\vec{D} = \frac{10y}{z} + 0 + 2 + 10\left(\frac{x^2y}{z^3}\right)$$

Put P(-2,3,5)

$$\nabla.\vec{D} = 6 + 2 + 0.96$$

$$\nabla.\vec{D} = 8.96 \text{ C/m}^3$$

9. Find the div and curl of D if $\vec{D} = x^2yi - (z^3 - 3x)j + 4y^2k$

$$\nabla.\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_Z}{\partial z}$$

$$\nabla.\vec{D} = \frac{\partial (x^2 y)}{\partial x} - \frac{\partial (z^3 - 3x)}{\partial y} + \frac{\partial 4y^2}{\partial z} = 2xy$$

$$\text{curl } \vec{D} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -(z^3 - 3x) & 4y^2 \end{vmatrix}$$

$$curl\,\vec{D} = i\left(\frac{\partial (4y^2)}{\partial y} - \frac{\partial (-(z^3 - 3x))}{\partial z}\right) - j\left(\frac{\partial (4y^2)}{\partial x} - \frac{\partial (x^2 y)}{\partial z}\right) + k\left(\frac{\partial (-(z^3 - 3x))}{\partial x} - \frac{\partial (x^2 y)}{\partial y}\right)$$

$$curl\,\vec{D} = (8y + 3z^2)i + (3 - x^2)k$$

10. Calculate the divergence of the vector field $\vec{F} = (3x + 2z^2)i + \frac{x^3 y^2}{z} j - (z - 7x)k$

$$\nabla.\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_Z}{\partial z}$$

$$\nabla.\vec{F} = \frac{\partial (3x + 2z^2)}{\partial x} + \frac{\partial \left(\frac{x^3 y^2}{z}\right)}{\partial y} + \frac{\partial \big(-(z - 7x)\big)}{\partial z}$$

$$\nabla.\vec{F} = 2 + \frac{2x^3 y}{z}$$

11. Write the position vector of a point P(2, -3,4) and hence find its magnitude and unit vector.

Solution:

Position vector $\vec{R} = 2\hat{a}_x - 3\hat{a}_y + 4\hat{a}_z$

Magnitude $|\vec{R}| = \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{29}$

Unit vector

$$\hat{a} = \frac{\vec{R}}{|\vec{R}|} = \frac{2\hat{a}_x - 3\hat{a}_y + 4\hat{a}_z}{\sqrt{29}}$$

12. Find the components and magnitude of $\vec{R}$ with initial point P (3,4,-1) and terminal point Q(4,7,3)

Solution:

Components are given by $(x_2 - x_1), (y_2 - y_1), and\ (z_2 - z_1)$

Components of $\vec{R}$ = (4 - 3),(7 – 4) and (3 – (-1))

Components of $\vec{R}$ = (1, 3, 4)

Magnitude of $\vec{R} = \sqrt{1^2 + 3^2 + 4^2} = \sqrt{26}$

13. Let $\vec{P} = (3, 2, 1)$ and $\vec{Q} = (4, 3, 7)$. Find $\overrightarrow{P + Q}, \overrightarrow{P - Q}\ and\ \overrightarrow{2P - Q}$. write the answers in component form

Solution:

$$\overrightarrow{P + Q} = (3 + 4, 2 + 3, 1 + 7) = (7,5,8)$$

Component form is $\overrightarrow{P+Q} = 7\hat{a}_x + 5\hat{a}_y + 8\hat{a}_z$

$\overrightarrow{P-Q} = (3-4, 2-3, 1-7) = (-1, -1, -6)$

Component form is $\overrightarrow{P-Q} = -1\hat{a}_x - 1\hat{a}_y - 6\hat{a}_z$

$\overrightarrow{2P-Q} = 2(3,2,1) - (4,3,7) = (6,4,2) - (4,3,7) = (2,1,-5)$

Component form is $\overrightarrow{2P-Q} = 2\hat{a}_x + 1\hat{a}_y - 5\hat{a}_z$

14. If the components of $\vec{P}, \vec{Q}$ and $\vec{R}$ are (3,2,4), (-1,4,7) and (5,2,-2) then show that $\left(\overrightarrow{P+Q}\right) + \vec{R} = \vec{P} + \left(\overrightarrow{Q+R}\right)$

Solution:

$$\overrightarrow{P+Q} = (3,2,4) + (-1,4,7) = (3-1, 2+4, 4+7) = (2,6,11)$$

$$\left(\overrightarrow{P\ +\ Q}\right) + \vec{R} = (2,6,11) + (5,2,-2) = (2+5, 6+2, 11-2)$$

$$\left(\overrightarrow{P\ +\ Q}\right) + \vec{R} = (7,8,9) - - - -(1)$$

$$\overrightarrow{Q\ \ +\ \ R} = (-1,4,7) + (5,2,-2) = (-1+5, 4+2, 7-2) = (4,6,5)$$

$$\vec{P} + \left(\overrightarrow{Q+R}\right) = (3,2,4) + (4,6,5) = (3+4, 2+6, 4+5)$$

$$\vec{P} + \left(\overrightarrow{Q+R}\right) = (7,8,9) - - - -(2)$$

From (1) and (2); $\left(\overrightarrow{P+Q}\right) + \vec{R} = \vec{P} + \left(\overrightarrow{Q+R}\right)$. Hence the proof.

15. Two forces acting on a body are $\overrightarrow{F_1} = 10\hat{a}_x - 20.4\hat{a}_y + 2\hat{a}_z$ and $\overrightarrow{F_2} = 15\hat{a}_x - 6.2\hat{a}_z$. Find the angle between them.

Solution:

$$\vec{F}_1.\vec{F}_2 = (10 \times 15) + (-20.4 \times 0) + (2 \times -6.2) = -162.4$$

$$|\vec{F}_1| = \sqrt{10^2 + 20.4^2 + 2^2} = 22.8$$

$$|\vec{F}_2| = \sqrt{15^2 + 6.2^2} = 16.2$$

$$cos\theta = \frac{\vec{F}_1.\vec{F}_2}{|\vec{F}_1||\vec{F}_2|} = \frac{-162.4}{22.8 \times 16.2} = -0.439$$

$$\theta = cos^{-1}(-0.439) = 116^0$$

Questions for exercise

1. Discuss scalar product and vector product along with examples
2. Explain the terms *scalar field* and *vector field*. Give examples
3. Explain scalar field in detail. Give examples
4. Explain in detail what vector field is. Give examples
5. Give expression for del operator. Is it a scalar or vector?
6. Define gradient of a scalar field. Give example. Write a note on its physical significance
7. What is divergence? Give the examples and expression for the same. discuss its physical significance
8. When does the field is said to be ***incompressible*** or ***solenoidal*** ?

9. Define curl of a vector field. Give the examples and expression for the curl. What is the value of curl for irrotational filed?
10. Discuss the significance of curl
11. Discuss three different types of integrations
12. State and explain Gauss divergence theorem
13. Prove the Gauss divergence theorem
14. State and explain Stokes' theorem
15. State and explain
    i. Gauss theorem in electrostatics
    ii. Gauss theorem in magnetism

    And hence write Maxwell's I and II equations
16. State and explain Biot-Sovart law
17. State and explain faraday's law of electromagnetic induction
18. State and explain Ampere's circuital theorem. What is its limitation?
19. Explain the Maxwell's correction to Ampere's circuital law.
20. What is displacement current? Obtain the expression for the same
21. Discuss the Maxwell's correction to Ampere's circuital theorem and hence write Ampere-Maxwell law
22. Distinguish between conduction current and displacement current
23. List the 4 Maxwell's equations for steady field in vacuum
24. Give a brief account of Maxwell's EM wave theory
25. Starting from Maxwell's equations derive the 2nd order differential equation for EM wave in terms of electric field
26. Starting from Maxwell's equations obtain the wave equation and velocity of EM waves in vacuum
27. Write a note on plane EM wave. Give the expression for Electric Field and Magnetic field variations
28. Give the properties of EM wave
29. What is polarization of wave? Mention different types of polarization of EM waves
30. Explain plane polarized, circularly polarized and elliptically polarized waves

* * * END * * *

MYSURU ROYAL INSTITUTE OF TECHNOLOGY

DEPARTMENT OF PHYSICS

Applied Physics for EEE Stream : 2022-23

Module – 5 : SEMICONDUCTORS & DEVICES

Syllabus:

Semiconductors & Devices: **8 hrs**

Fermi level in Intrinsic & Extrinsic Semiconductor, Expression for concentration of electrons in conduction band & holes concentration in valance band (only mention the expression), Relation between Fermi energy & Energy gap in intrinsic semiconductors (derivation), Law of mass action, Electrical conductivity of a semiconductor (derivation), Hall effect, Expression for Hall coefficient (derivation) and its application. Photo-diode and Power responsivity, Construction and working of Semiconducting Laser, Four probe method to determine resistivity, Phototransistor, Numerical problems.

Pre-requisite: Basics of Semiconductors

Self-learning: Fermi level in Intrinsic & Extrinsic Semiconductor

SEMICONDUCTORS

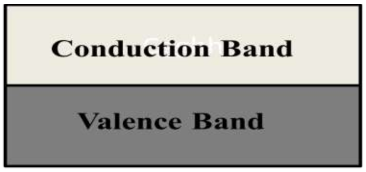
Introduction:

Band theory of solids: Band theory of solids can explain the conductivity of conductors, insulators and semiconductors on energy gap between the bands.

Classification of solids into conductors, insulators and semiconductors

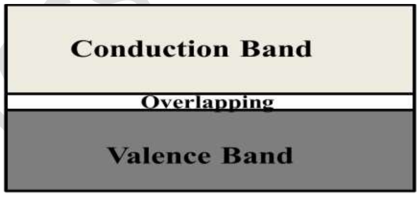
Based on electrical properties, solids can be classified as conductors, insulators and semiconductors. The order of magnitude of electrical conductivity serves as an indicator for the classification. There are few exceptions like conductors with incomplete inner orbits (like Sn, Mn, Zr, Rh etc.,) showing higher resistivity and narrow band gap semiconductors (like InSb, Bi_2Te_3 etc.,) with exceptionally high conductivity. Hence conductivity alone should not be the basis for classification of solids. The temperature coefficient of resistivity helps us to classify and distinguish between highly conducting semiconductors and less conducting metals. The positive value of temperature coefficient of resistivity for conductors and negative value for semiconductors can be explained on the basis of band theory.

Conductors: These are the solids which really conduct electricity. There are two types of energy bands in conductors depending on the electronic configuration of atoms.



Energy band in conductors

or

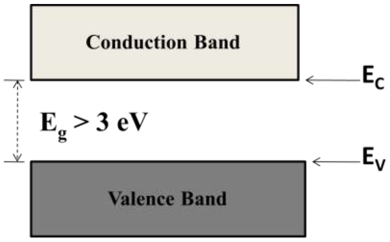


Energy band in conductors

In alkali metals and other metals having configuration ns^1 or ns^2np^1 etc. having unpaired electrons in the outermost orbit of their atoms and hence the valence band is partially filled. So that, the electrons are easily excited to the higher levels in the same band. As a very large number of vacant levels exists, a large current can flow in conductors.

In conductors having paired electrons in the outermost orbit the valence band is completely filled. So they should not conduct electric current. But it is observed that they also conduct electric current. This is because conduction band overlap with the valence band forming a composite band which is also partially filled as in the earlier case. In these conductors the forbidden gap/energy gap $E_g = 0$.

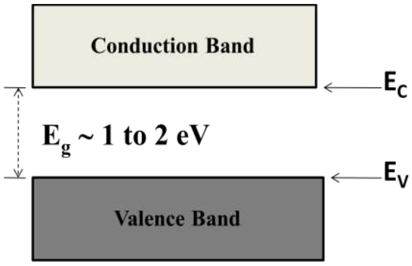
Insulators: These are solids, which don't conduct electric current. In insulators valence band is completely filled and conduction band is completely empty. They are separated by a very wide energy gap of the order of 6 eV (> 3 eV).



Energy Band in Insulator

Since the valence band is completely filled, the electrons can't move, so they cannot conduct electricity. The conduction band is completely empty having no electrons to move. To excite electrons from valence band to conduction band, we require an electric field of the order of $6x10^8$ V/m. hence an insulator doesn't conduct electric current under normal condition.

Semiconductors: These are the solids, having conductivity less than the conductors and greater than the insulators. The energy band diagram of a semiconductor is similar to that of an insulator but, the valence band and conduction band are separated by a small narrow forbidden energy gap of the order of < 3 eV.



Energy Band in Semiconductor

Theory of semiconductors

There are two types of semiconductors; pure and impure. The pure semiconductors are called intrinsic semiconductors and the impure semiconductors are called extrinsic semiconductors.

Extrinsic semiconductors are obtained by adding very small impurities to pure semiconductors in a controlled manner. This process is called "doping" and the impurity added is called "dopant". Doped semiconductors are called 'extrinsic semiconductors'. By virtue of small impurity in an extrinsic semiconductor, it posses high electrical conductivity. There are two types of extrinsic semiconductors are there. They are

1. n - type semiconductors and
2. p - type semiconductors.

i. **n - type semiconductors:** If a donor impurity (pentavalent atoms like As, Sb) are added to a tetravalent pure semiconductor such as germanium or silicon the – ve charge carriers are more than the + ve charge carriers. These types of semiconductors are called – ve type or n - type semiconductors.

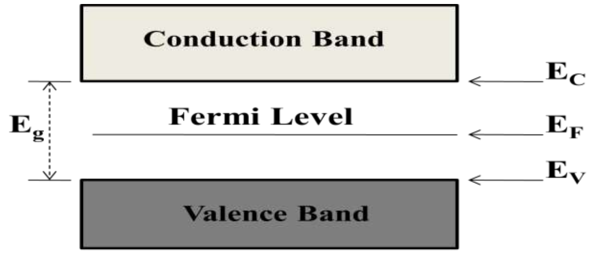
ii. **p - type semiconductors:** If an acceptor impurity (trivalent atoms like B) are added to pure semiconductors, the +ve charge carriers are more than the – ve charge carriers. These types of semiconductors are called + ve type or p-type semiconductors.

Further due to thermal excitation, electron-hole pair is formed by the breaking of bonds, which become additional charge carriers. Thus both type charge carriers are observed in extrinsic semiconductors. However, the type of charge carriers which owe their formation due to only the thermal excitation and not due to doping are very small in number and they are called 'minority carriers'. The other type charge carriers obtained due to both doping as well as thermal excitation are called 'majority carriers'.

Significance of Fermi level

According to Fermi-Dirac statistics, at temperature T = 0 K, all the energy levels below Fermi level are completely filled and the ones above are completely empty. Thus Fermi level (E_F) at 0 K acts as a distinguished energy position between the filled and unfilled energy states.

Fermi level in intrinsic semiconductor

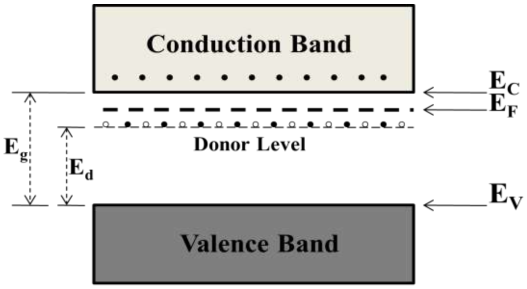


Fermi level in Intrinsic Semiconductor

Energy gap (E_g) is the difference between the bottom of the conduction band (E_c) and top of the valence band (E_v) i.e., $E_g = E_c - E_v$. For convention, the energy at the top of the valence band is taken as zero (i.e., $E_v = 0$) for reference. At T = 0 K, all the energy levels in the valence band are completely filled and all energy levels in the conduction band are completely empty. But at room temperature, due to thermal excitation, some of the electrons at the top of the valence band are able to jump the energy gap and occupy some energy levels at the bottom part of the conduction band. These electrons return soon to the vacant energy levels left in the valence band. The electrons in this set of energy levels continue to undergo excitation and de-excitation, and thus becomes conduction electrons. Based on this one can say that, conduction electrons between the energy levels in the bottom part of conduction band and the top portion of the valence band. Due to this distribution, the average energy of the electrons taking part in conduction will be almost equal to $(E_c + E_v)/2 = E_g/2$ (since $E_v = 0$ & $E_c = E_g$). Thus Fermi level lies in the mid part of the forbidden gap for an intrinsic semiconductors.

Impurities level and Fermi level in extrinsic semiconductor

a) n - type semiconductors :

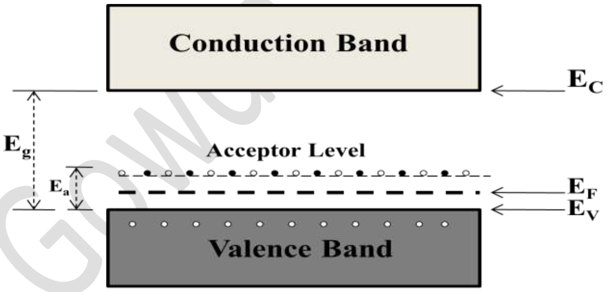


Fermi level in n - type Semiconductor

In case of n-type semiconductors, the electrons of donor atoms are free for the movement as compared to the resistance of electrons in that material, and hence they possess more energy. As a result the energy level for the electrons will be elevated to a position much higher than the valence band and lie very close to the conduction band as shown in the figure. These levels are called donor levels. The energy difference between the top of the valence band and the donor level is denoted as E_d. At low temperatures, thermal energy will be less and hence electrons can't move from valence band to conduction band. But the same energy will be enough to transfer the electrons from donor level to the conduction band (since, $E_g - E_d$ is small). These electrons are mainly responsible for conduction in n - type materials. Since these electrons occupy the energy levels very close to the bottom of the conduction band, the electrons get distributed between the energy levels in the bottom part of the conduction band and the donor levels. The difference between these two states will be almost equal to $E_c - E_d$ (or $E_g - E_d$). Thus the average energy of electrons participating in conduction becomes

$\frac{1}{2}(E_c + E_d)$ or $\frac{1}{2}(E_g + E_d)$, and hence Fermi level in an n-type material at low temperature will be located in the forbidden band at the level $\frac{1}{2}(E_c + E_d)$ or $\frac{1}{2}(E_g - E_d)$ just below the bottom of conduction band.

b) p - type semiconductors:



Fermi level in p - type Semiconductor

In case of p - type semiconducting materials, the acceptor atoms give rise to holes which are relatively free compared to those which are in the conduction band. Hence holes due to acceptor atoms possess higher energy than those in the conduction band. Since the energy for holes increases in the downward direction in the band diagram & they occupy energy levels in the band gap close to the valence band as shown in figure. These energy levels referred as acceptor levels. The energy difference between the acceptor levels & the top of the valence band is denoted by $E_a - E_v = E_a$ (since $E_v = 0$). Since E_a is small, a small amount of thermal energy is sufficient to transfer the holes from the acceptor levels to the valence band. At low temperature, the holes participate in the conduction undergo excitation and de-excitation between the acceptor level and energy levels in the top portion of the valence band. As a result, the average energy of the holes participating in conduction becomes (E_a + E_v)/2 or E_a/2 (since E_v = 0). Hence, the Fermi level in p-type material at low temperature will be located in the forbidden gap at the level (E_a + E_v)/2 or E_a/2 from the top of the valence band.

Carrier concentration in an intrinsic semiconductor

1) Expression for concentration (density) of electrons.

The number of electrons present in the conduction band of an intrinsic semiconductor per unit volume is known as Electron concentration. It is given by

$$N_e = 2\left(\frac{2\pi m_e^* kT}{h^2}\right)^{\frac{3}{2}} \times e^{\left(\frac{E_F - E_g}{kT}\right)} \qquad \rightarrow (1)$$

$$or \quad N_e = \frac{4\sqrt{2}}{h^3}\left(\pi m_e^* kT\right)^{\frac{3}{2}} \times e^{\left(\frac{E_F - E_g}{kT}\right)} \qquad \rightarrow (1)$$

This is also the expression for electron concentration.

2) Expression for hole concentration:

The number of holes present in the valence band of an intrinsic semiconductor per unit volume is known as hole concentration. It is given by

$$N_h = 2\left(\frac{2\pi m_h^* kT}{h^2}\right)^{\frac{3}{2}} \times e^{-\left(\frac{E_F}{kT}\right)} \qquad \rightarrow (2)$$

$$or \quad N_h = \frac{4\sqrt{2}}{h^3}\left(\pi m_h^* kT\right)^{\frac{3}{2}} \times e^{-\left(\frac{E_F}{kT}\right)} \qquad \rightarrow (2)$$

This is also the expression for hole concentration.

Relation between Fermi energy and energy gap for an intrinsic semiconductor (or Fermi energy in intrinsic semiconductor)

For an intrinsic semiconductor, the number of electrons per unit volume in conduction band is equal to the number of holes per unit volume in valence band.

$$\therefore \quad N_e = N_h$$

By substituting the values of N_e and N_h we get

$$2\left(\frac{2\pi m_e^* kT}{h^2}\right)^{\frac{3}{2}} \times e^{-\left(\frac{E_c-E_F}{kT}\right)} = 2\left(\frac{2\pi m_h^* kT}{h^2}\right)^{\frac{3}{2}} \times e^{\left(\frac{E_V-E_F}{kT}\right)}$$

$$or \quad \left(m_e^*\right)^{\frac{3}{2}} \times e^{-\left(\frac{E_c-E_F}{kT}\right)} = \left(m_h^*\right)^{\frac{3}{2}} e^{\left(\frac{E_V-E_F}{kT}\right)}$$

$$e^{\left(\frac{-E_c+E_F-E_V+E_F}{kT}\right)} = \left(\frac{m_h^*}{m_e^*}\right)^{\frac{3}{2}}$$

$$\text{OR} \quad e^{\left(\frac{2E_F-(E_C+E_V)}{kT}\right)} = \left(\frac{m_h^*}{m_e^*}\right)^{\frac{3}{2}}$$

Taking logarithm on both sides, we get

$$\frac{2E_F-(E_C+E_V)}{kT} = \frac{3}{2}\log_e\left(\frac{m_h^*}{m_e^*}\right)$$

$$\text{OR} \quad 2E_F-(E_C+E_V) = \frac{3}{2}kT\log_e\left(\frac{m_h^*}{m_e^*}\right)$$

$$E_F-\left(\frac{E_C+E_V}{2}\right) = \frac{3}{4}kT\log_e\left(\frac{m_h^*}{m_e^*}\right)$$

$$E_F = \frac{E_C+E_V}{2}+\frac{3}{4}kT\log_e\left(\frac{m_h^*}{m_e^*}\right)$$

$$OR, \quad E_F = \frac{E_C+E_V}{2}-\frac{3}{4}kT\log_e\left(\frac{m_e^*}{m_h^*}\right)$$

At T = 0 K, $E_F = \left(\frac{E_C+E_V}{2}\right)$ This is the expression for Fermi energy.

As we know that, $E_v = 0$ & $E_c = E_g$, then

$$E_F = \frac{E_g}{2}$$

Thus the Fermi level is in the middle of the band gap for an intrinsic semiconductor.

Intrinsic carrier concentration

In an intrinsic semiconductor there will be continuous production of electron-hole pairs due to thermal energy. At the same time there will be decrease in the number of electrons and holes due to electron-hole combination. As long as temperature remains constant the electron-hole concentration will also remain constant. Thus at temperature T K, $N_e = N_h = N_i$. Where N_i is called intrinsic charge concentration.

Therefore the product of concentrations gives; $N_i^2 = N_e \times N_h$.

On substituting for N_e & N_h in the above equation, we get

$$N_i = 2\left(\frac{2\pi kT}{h^2}\right)^{\frac{3}{2}} \left(m_e^* m_h^*\right)^{\frac{3}{4}} e^{-\left(\frac{E_g}{2kT}\right)} \quad \rightarrow (1)$$

Equations (1) represents charge carrier concentrations in an intrinsic semiconductor.

Conductivity in intrinsic semiconductor:

Conductivity will take place in an intrinsic semiconductor due to the movement of electrons and holes (charge carriers) under the application of electric field. The average velocity acquired by the charge carriers in the presence of electric field E, is called drift velocity V_d.

The drift velocity, $V_d \propto E$ or $V_d = \mu E \quad \rightarrow (1)$

Where μ is called mobility of charge carriers. It is defined as the velocity acquired by a carrier per unit electric field. i.e. $\mu = \frac{V_d}{E} \quad \rightarrow (2)$

The net transfer of charges through the cross-section of the specimen gives the electric current in it. The total electric current (I) in the semiconductor specimen is the sum of the current due to drifting of electrons in the conduction band (I_e) and the current due to drifting of holes in the valence band(I_h).

$$\therefore I = I_e + I_h \qquad \rightarrow (3)$$

To find an expression for electrical conductivity, first we shall consider the flow of electrons in the semiconductor. Let 'A' be the area of cross-section, N_e is the number of electrons/unit volume and 'e' is the magnitude of charge of an electron, then the flow of electron/second in the conduction band gives the electron current I_e.

$$\therefore I_e = N_e e A v_{de} \qquad \rightarrow (4)$$

∴ Current density of electrons is given by,

$$J_e = \frac{I_e}{A} = N_e e v_{de}$$

$$J_e = N_e e \mu_e E \qquad \rightarrow (5) \quad \text{Since} \quad v_{de} = \mu_e E$$

But ohm's law gives the relation for $\mathbf{J_e}$ as $J_e = \sigma_e E \quad \rightarrow (6)$

On comparing equations (5) & (6), we get $\sigma_e = N_e e \mu_e \qquad \rightarrow (7)$

Is the electronic conductivity is the conduction band. Similarly, the conductivity due to holes in the valence band is given by

$$\sigma_h = N_h e \mu_h \qquad \rightarrow (8)$$

Where, N_h and μ_h represents the concentration & mobility of holes respectively. The total conductivity of the semiconductor is given by,

$$\sigma = \sigma_e + \sigma_h = N_e e \mu_e + N_h e \mu_h$$

$$\text{or}\quad \sigma = e\left(N_e\mu_e + N_h\mu_h\right) \qquad \rightarrow (9)$$

In an intrinsic semiconductor, $N_e = N_h = N_i$ is the intrinsic carrier density, so that equation (9) becomes

$$\sigma_i = N_i e\left(\mu_e + \mu_h\right) \qquad \rightarrow (10)$$

Using the expression for N_i from intrinsic semiconductor

i.e., $N_i = 2\left(\frac{2\pi k}{h^2}\right)^{\frac{3}{2}} \times \left(m_e^* \times m_h^*\right)^{\frac{3}{4}} T^{\frac{3}{2}} \times e^{-\left(\frac{E_g}{2kT}\right)}$ in equation (10), we get

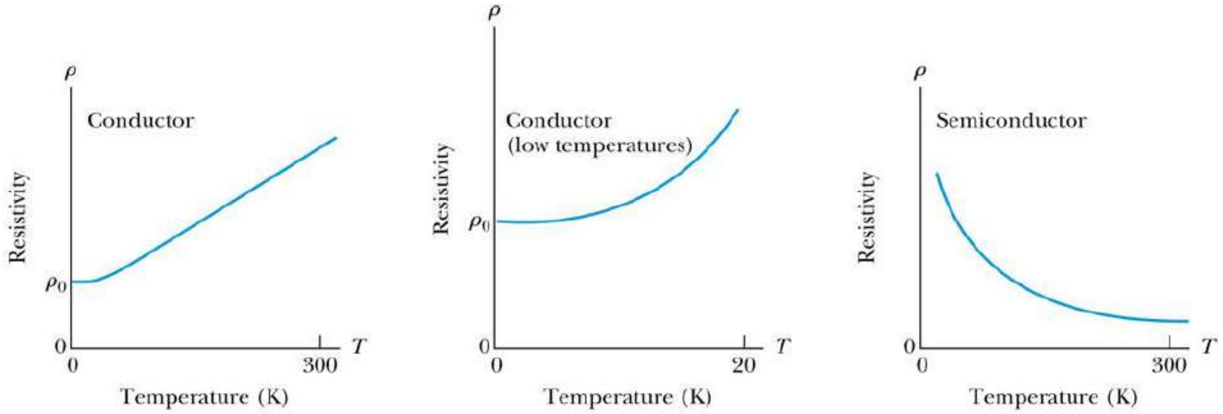
$$\sigma_i = 2\left(\frac{2\pi k}{h^2}\right)^{\frac{3}{2}} \times \left(m_e^* \times m_h^*\right)^{\frac{3}{4}} \times T^{\frac{3}{2}} \times e\left(\mu_e + \mu_h\right) \times e^{-\left(\frac{E_g}{2kT}\right)}$$

$$\text{or}\quad \sigma_i = \sigma_o e^{-\left(\frac{E_g}{2kT}\right)} \qquad \rightarrow (11)$$

where $\sigma_o = 2\left(\frac{2\pi k}{h^2}\right)^{\frac{3}{2}} \times \left(m_e^* \times m_h^*\right)^{\frac{3}{4}} \times T^{\frac{3}{2}} \times e\left(\mu_e + \mu_h\right)$

Equation (10) represents the expression for electrical conductivity in intrinsic semiconductor.

Variation of Resistance or Resistivity with temperature in intrinsic semiconductor :



Resistance of any material is varies with temperature. The resistance of metals increases with temperature, whereas the resistance of the semiconductors is decreases with increase of temperature. The expression for electrical conductivity in an intrinsic semiconductor is given by

$$\sigma_i = \sigma_o e^{-\left(\frac{E_g}{2kT}\right)} \qquad \rightarrow (1)$$

Where, $\sigma_o = 2\left(\frac{2\pi k}{h^2}\right)^{\frac{3}{2}} \times \left(m_e^* \times m_h^*\right)^{\frac{3}{4}} \times T^{\frac{3}{2}} \times e\left(\mu_e + \mu_h\right)$

Since, $\rho = \frac{1}{\sigma}$

$$\rho_i = \frac{1}{\sigma_i} = \frac{1}{\sigma_o} e^{\left(\frac{E_g}{2kT}\right)} \quad \rightarrow (2)$$

This is the expression for resistivity of the semiconductor.

Since $\rho = \frac{RA}{L}$, $\quad or; \quad \frac{RA}{L} = \frac{1}{\sigma_o} e^{\left(\frac{E_g}{2kT}\right)}$

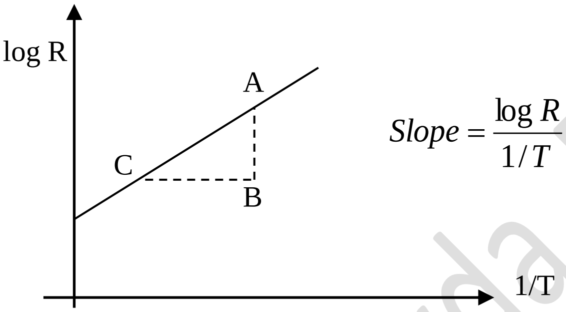
$$or, \quad R = \frac{L}{A\sigma_o} e^{\left(\frac{E_g}{2kT}\right)} = B\, e^{\left(\frac{E_g}{2kT}\right)}$$

$$or, \quad R = B\, e^{\left(\frac{E_g}{2kT}\right)} \qquad where \quad B = \frac{L}{A\sigma_o}$$

Taking logarithm on both sides we get,

$$\log R = \log B + \frac{E_g}{2kT} \qquad or \quad \log R = \log B + \left(\frac{E_g}{2k}\right)\frac{1}{T} \qquad (3)$$

The above equation (3) represents the equation for a straight line in which $E_g/2k$ is the slope. A graph of logR and 1/T is plotted.



In the curve slope gives $\frac{\log R}{1/T} = \frac{E_g}{2k}$.

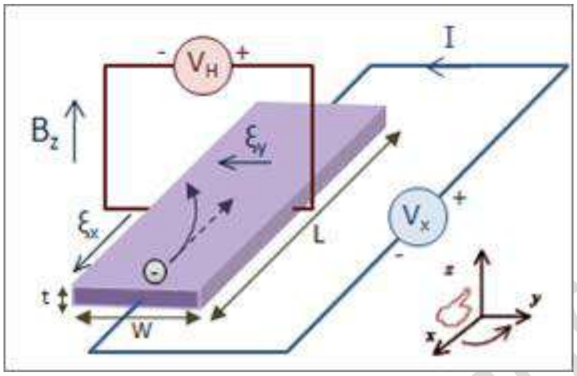
$$E_g = 2k\left(\frac{\log R}{1/T}\right) \qquad (4)$$

This is the expression for the energy gap of a semiconductor in joules.

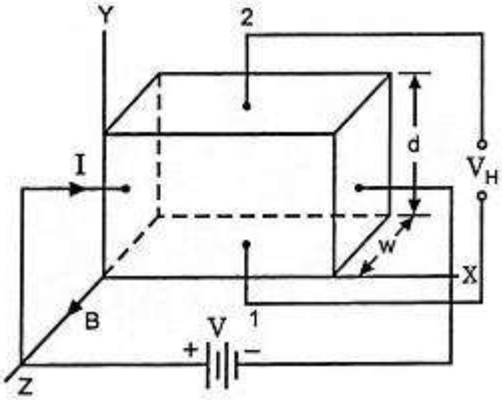
Hall Effect:

"When a current carrying conductor is place in a magnetic field, a voltage will be generated perpendicular to both the current and the magnetic field. This principle is known as the hall effect."

This was observed first by E. H. Hall in the year 1879. The potential difference developed in a direction perpendicular to both current and magnetic field is called Hall voltage and the corresponding electric field is called Hall field.



Consider a conductor carrying current along +ve X axis. Magnetic field is applied along +ve Z axis. Thus electrons experience Lorentz force along the -ve Y axis. Thus electrons accumulate on the lower surface which results in the accumulation of +ve charges on the upper surface. This results in an electric field developed between the two surfaces.



The force on the electrons due to electric field further opposes the force due to magnetic field.

A stage is reached where in both the forces are equal and an equilibrium state is reached The electric field developed across the material at equilibrium is called Hall Field.

Expression for Hall Co-efficient:

The Lorentz force exerts on electrons by the influence of magnetic field along +ve Y – axis is given by

$$F_L = Bev \quad \rightarrow (1)$$

The force exerts on electrons by the induced electric field along -ve Y – axis is given by

$$F_E = eE_H \quad \rightarrow (2)$$

Under equilibrium the Lorentz force on electrons is equal to force due to hall field. i.e., $F_L = F_E$

$$\therefore \quad eE_H = Bev \quad \rightarrow (3)$$

Here e is electronic charge, E_H is Hall Field, B is Magnetic field, v is the velocity of electrons. The current density J is given by

$$J = nev \quad \rightarrow (4)$$

Here n is the number density of charges. Dividing equation (3) by (4) we get

$$\frac{eE_H}{J} = \frac{Bev}{nev}$$

$$\frac{eE_H}{J} = \frac{B}{n}$$

$$E_H = \frac{BJ}{ne}$$

$$E_H = R_H BJ$$

Here R_H is called the Hall coefficient and it is given by

$$R_H = \frac{1}{ne}$$

The Hall voltage is given by the equation

$$V_H = E_H d = R_H BJd$$

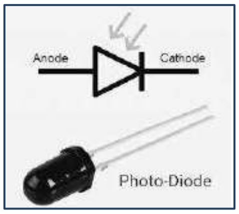
Here d is the thickness of the material along y-axis.

Note: In case of n-type semiconductors R_H is negative and for p-type semiconductors R_H is positive.

Application of Hall Effect

a. This effect is the basis of many practical applications and devices such as magnetic field measurements, and position and motion detectors.
b. Hall Effect is also used in the determination of type of extrinsic semiconductor (P-Type and N-Type).
c. Hall Effect is also used to find semiconductor properties like carrier concentration and mobility of carriers.

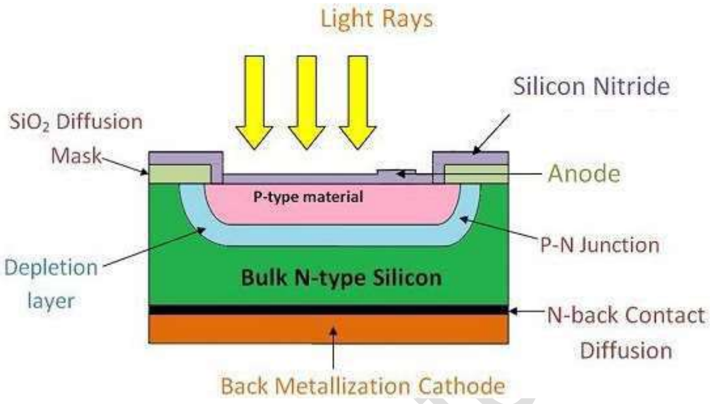
Photodiode:



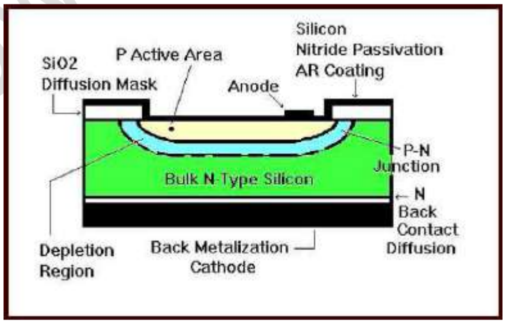
Photodiode is a two terminal electronic device which, when exposed to light the current starts flowing in the diode. It is operated in reverse biased mode only. It converts light energy into electrical energy. When the ordinary diode is reverse biased the reverse current starts increasing with reverse voltage the same can be applied to the photodiode. But in the case of photodiode the current can flow without application of reverse voltage, the P-N junction of the photodiode is illuminated by light and light energy dislodge valence electrons and the diode starts conducting.

They are also called a photo-detector, a light detector, and a photo-sensor. Photodiodes are designed to work in reverse bias condition.

Construction of Photodiode:

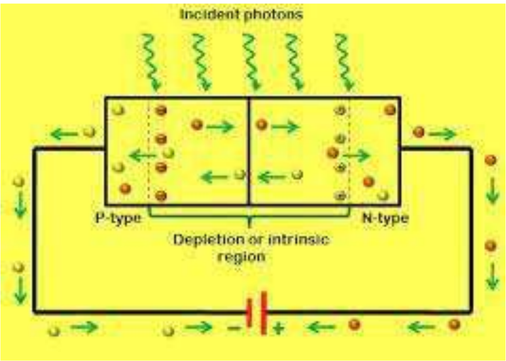


The photodiode is made up of two layers of P-type and N-type semiconductor. In this, the P-type material is formed from diffusion of the lightly doped P-type substrate. Thus, the layer of P+ ions is formed due to the diffusion process. And N-type epitaxial layer is grown on N-type substrate. The P+ diffusion layer is developed on N-type heavily doped epitaxial layer. The contacts are made up of metals to form two terminal cathode and anode.



The front area of the diode is divided into two types that are active surface and non-active surface. The non-active surface is made up of SiO2 (Silicon di Oxide) and the active surface is coated with anti-reflection material. The active surface is called so because the light rays are incident on it. While on the non-active surface the light rays do not strike. The active layer is coated with anti-reflection material so that the light energy is not lost and the maximum of it can be converted into current. The entire unit has dimensions of the order of 2.5 mm.

Working of Photodiode:



The working principle of a photodiode is, when a photon of ample energy strikes the diode, it makes a couple of an electron-hole. This mechanism is also called the inner photoelectric effect. If the absorption arises in the depletion region junction, then the carriers are removed from the junction by the inbuilt electric field of the depletion region.

Therefore, holes in the region move toward the anode, and electrons move toward the cathode, and a photocurrent will be generated. The entire current through the diode is the sum of the absence of light and the photocurrent. So the absent current must be reduced to maximize the sensitivity of the device.

Responsivity is a measure of input output gain of the detector. It is the measure of electrical output per optical input. The responsivity has units of amperes per watt.

$$R_\lambda = I_P / P$$

Where, R_λ = Responsivity

I_P = Output Photo Current

P = Incident Optical Power

Application of Photodiode:

The photodiode is used in optical communication system.

The photodiode is used in automotive devices.

The photodiode is used in medical devices.

It is used in solar cell panels.

The Photodiode are used in consumer electronics devices.

It is used for exact measurement of the intensity of light in science & industry.

It is used in character recognition circuit.

It is used in camera light meters, and street lights.

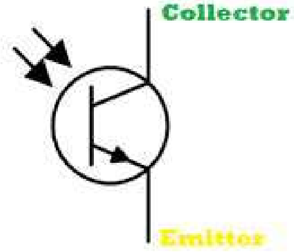
It is used in demodulation.

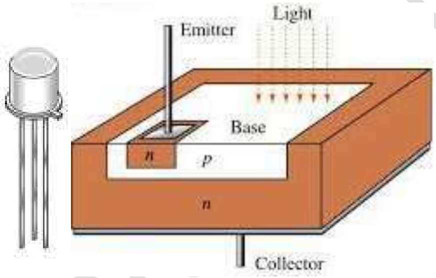
The photodiode is used in logic circuit.

It is used in photo detection circuits.

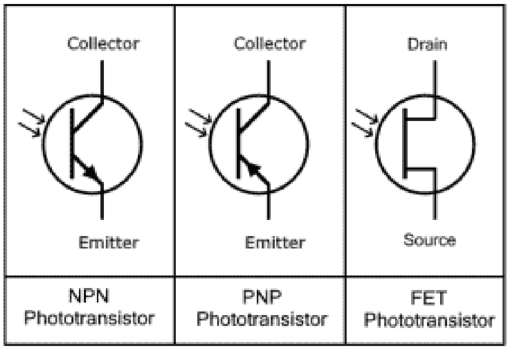
Phototransistor:

A phototransistor is a light controlled switch that switches a circuit and amplifies the current when exposed to light It is a three layer semiconductor device whose light sensitive base is exposed The light striking the base converts into a base current that amplifies the current between the emitter and collector proportional to the intensity of light They are used for sensing light pulses of high speed and small magnitude It is similar to BJT except for the exposed base instead of a terminal

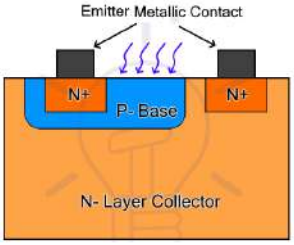




Unlike a normal transistor a phototransistor has only two terminals, an emitter and collector The symbol of the phototransistor is very similar to any normal transistor except for the base terminal Instead of the base terminal, there are two pointing arrows representing incident light.



It has a similar design to a normal bipolar transistor The base and collector region is larger as compared to a normal transistor And the base is covered with transparent epoxy resin and lens to prevent the base from contamination as well as focus light when entering the region Physically it resembles a photodiode.



The emitter is heavily doped as compared to the collector but the collector is relatively very large The physical area of the base and collector region is kept large to collect more light The larger area generates more base current, that is amplified Thus making it more sensitive than a photodiode.

Working:

Phototransistor operates just like any normal bipolar transistor except for the fact that the base current is generated by a light source instead of a voltage source The base current is generated on the principle of the photovoltaic effect According to this phenomenon when photons strike the PN junction, electron hole pairs are generated that separate and move in the opposite direction thus creating a base current The base current is then amplified by the transistor action Therefore the Phototransistor is 100 times more sensitive than the photodiode.

When biasing, the collector is kept at a higher voltage with respect to the emitter in NPN phototransistor while in PNP, the collector is at a lower voltage with respect to the emitter And the collector to the base junction is reverse biased the base terminal (in the case of three lead phototransistors) is kept open or not connected, otherwise, it will operate as a normal transistor.

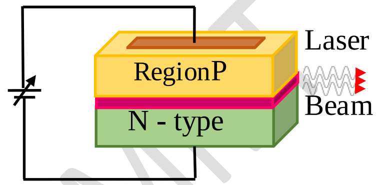
Under no light conditions, there is a small reverse saturation current or leakage current called dark current that is directly proportional to the temperature as in photodiodes When light shines on the phototransistor, the lens focuses the light onto the collector base junction and generates a base current due to the photovoltaic effect The base current is amplified hundreds of times.

Application of Phototransistors

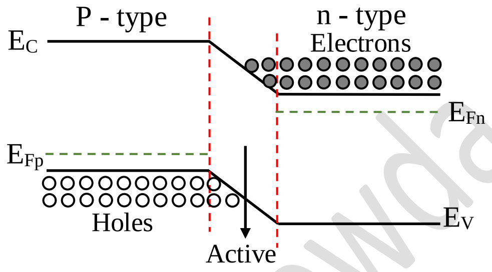
For light detecting and controlling
In counting systems and punch card readers
In relays
Alarm Systems
Level Indicators
Proximity Detectors
Encoders.

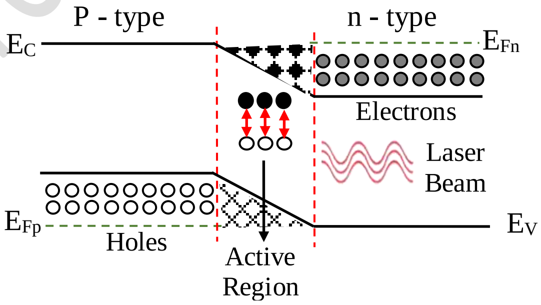
Semiconductor laser:

Principle: A semiconductor diode laser is a specially fabricated P-N junction device that emits coherent light when it is forward biased. It works on the principle of LED where p-type and n-type semiconductors are heavily doped. Hence electrons and holes are recombined in the depletion region producing coherent beam radiations.



Construction: The schematic diagram of semiconductor diode laser is shown in figure. It consists of a heavily doped n and p regions. The n-region is obtained by doping with pentavalent tellurium and the p-region is obtained by doping with trivalent zinc. The p-n region lies in a horizontal plane through the centre. The top and bottom faces of a diode are metalized to pass current through the diode. The front and rear faces are well polished parallel to each other and perpendicular to the plane of the junction. The other two opposite faces are roughened to prevent the lasing action in that direction. The first figure indicates energy level diagram for ordinary or LED diode and the second figure indicates the energy level diagram for Laser diode.





Working: The Diode is forward biased using an external source. Therefore, electrons and holes flow across the junction. The injected electrons and holes in the depletion region cause spontaneous emission of photons and the junction acts as LED. A population inversion is achieved in the depletion region of heavily doped P-N junction semiconductor diode in forward biased. Hence more electrons are occupied in donor levels and conduction band of n-type semiconductor, and the Fermi level lies within the conduction band. Similarly, the acceptor levels are unoccupied and more holes are existing in the valence band and the Fermi level lies within the valence band.

When a diode is forward biased, the energy levels shifted and the new distribution is as shown in 2nd figure. As the current is increased, the intensity of light increased. When the current reaches a threshold value the carrier concentration in the depletion region will reach very high values. This region contains a large concentration of electrons in CB and holes in VB. i.e., upper levels in the depletion region are having high population while the lower levels are vacant. This is the population inversion. The narrow region where the state of population inversion is achieved is called active region. The Stimulated electron - hole recombination cause emission of coherent beam of radiation. At room temperature GaAs semiconductor diode emits laser light of wavelength 9000Å in IR region. A GaAsP emits laser of wavelength 6500Å in the visible region as red light.

The semiconductor diode lasers are simple, compact and highly efficient. They require very little power. Diode lasers give more divergent beam having an angular spread of the order of 5°-15°. They are less monochromatic and highly temperature sensitive. In semiconductor diode there is no meta-stable state.

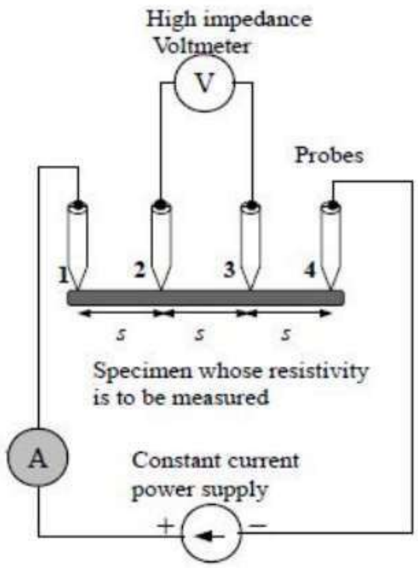
Advantages

1. It has excellent efficiency
2. The output can be modulated
3. It produces both continuous wave output or pulsed output.
4. It is highly economical

Applications

1. It is used in optical fiber communication.
2. It is used in commercial CD recording and reading.

Four probe method:



The four probe apparatus is one of the standard and most widely used apparatus for the measurement of resistivity of semiconductors The apparatus consists of four equally spaced tungsten metal tips with finite radius) Each tip is supported by springs on the end to minimize sample damage during probing The four metal tips are part of an auto mechanical stage which travels up and down during measurements A high impedance current source is used to supply current through the outer two probes, a voltmeter measures the voltage across the inner two probes to determine the sample resistivity Typical probe spacing is around 2 mm These inner probes draw no current because of the high input impedance voltmeter in the circuit Thus unwanted voltage drop at point 2 and point 3 caused by contact resistance between probes and the sample is eliminated from the potential measurements.

In order to use this four probe method in germanium crystals or slices it is necessary to assume that

1. The resistivity of the material is uniform in the area of measurement.
2. A non-conducting boundary is produced when the surface of the crystal is in contact with an insulator.

Case 1 Resistivity Measurements on a Large Sample.

We assume that the metal tip is infinitesimal and sample is semi-infinite in lateral dimensions. For bulk samples where the sample thickness, W >> S, the probe spacing, we assume a spherical protrusion of current emanating from the outer probe tips. The resistivity is computed to be

$$\rho_0 = \frac{V}{I} x 2\pi S$$

V = floating potential difference between the inner probes

I = Current through the outer pair of probes

S = Spacing between point probes

ρ_0 = Resistivity of the sample

Case 2: Resistivity Measurements on a Thin Slice

For the case of a non-conducting bottom on a slice the resistivity is computed from

$$\rho = \frac{\rho_0}{f\left(W/S\right)}$$

Case -3: Temperature dependence of resistivity

The resistivity ρ of a semiconductor depends on the temperature T as follows

$$\rho = A \exp \frac{E_g}{2kT}$$

where A is a constant, E_g is the energy gap of the semiconductor.

Problems :

1. The mobility of electrons and holes in a sample of intrinsic germanium at 300K are 0.36 m^2/Vs and 0.14 m^2/Vs respectively. If the resistivity of the specimen is 2.2 Ω.m, compute the intrinsic concentration.

Solutions,

$\mu_e = 0.36\ m^2/Vs$

$\mu_h = 0.14\ m^2/Vs$

$\rho_i = 2.2\ \Omega m$

$n_i = ?$

we know that,

$$\sigma_i = N_i e(\mu_e + \mu_h)$$

Or $$\rho_i = \frac{1}{\sigma_i} = \frac{1}{n_i e(\mu_e + \mu_h)}$$

$$\therefore n_i = \frac{1}{\rho_i e(\mu_e + \mu_h)} = \frac{1}{2.2 \times 1.602 \times 10^{-19}(0.36 + 0.14)}$$

$$n_i = 5.67 \times 10^{18} / m^3$$

2. The following data are given for intrinsic germanium at 300 K. $n_i = 2.4 \text{x} 10^{19}/m^3$, $\mu_e = 0.39\ m^2/Vs$, $\mu_h = 0.19\ m^2/Vs$. calculate the resistivity of the sample.

Solutions,

$n_i = 2.4 \text{x} 10^{19}/m^3$

$\mu_e = 0.39\ m^2/Vs$

$\mu_h = 0.19\ m^2/Vs$

$\rho_i = ?$

we know that,

$$\rho_i = \frac{1}{n_i e(\mu_e + \mu_h)} \qquad \because \rho_i = \frac{1}{\sigma_i}$$

$$\rho_i = \frac{1}{2.4 \times 10^{19} \times 1.602 \times 10^{-19}(0.39 + 0.19)} = 0.448 \quad \Omega m$$

3. Find the conductivity of intrinsic silicon at 300 K. It is given that n_i at 300 K in silicon is $1.5 \text{x} 10^{16}/m^3$ and mobilities of electrons and holes in silicon are 0.13 m^2/Vs and 0.05 m^2/Vs respectively.

Solution,

$n_i = 1.5 \text{x} 10^{16}/m^3$

$\mu_e = 0.13\ m^2/Vs$

$\mu_h = 0.05\ m^2/Vs$

we know that,

$$\boldsymbol{\sigma_i = n_i e (\mu_e + \mu_h)}$$

$$\sigma_i = 1.5 \text{x} 10^{16} \text{x} 1.602 \text{x} 10^{19} (0.13+0.05)$$

$$\sigma_i = 4.325 \text{x} 10^{-4} / \Omega m$$

4. For an intrinsic semiconductor with gap width $E_g = 0.8$ eV, calculate the concentration of intrinsic charge carriers at 300 K assuming that $m_n^* = m_p^* = m_e$ (rest mass of electrons).

Solutions,

$E_g = 0.8$ eV

T=300 K

$N_i = ?$

we know that,

$$N_i = 2\left(\frac{2\pi kTm_e}{h^2}\right)^{\frac{3}{2}} e^{-\left(\frac{E_g}{2kT}\right)} \qquad \because m_e^* = m_p^* = m_e$$

$$\therefore N_i = 2\left(\frac{2\pi km_e}{h^2}\right)^{\frac{3}{2}} \times T^{\frac{3}{2}} e^{-\left(\frac{E_g}{2kT}\right)}$$

$$N_i = 2\left(\frac{2\pi \times 1.38 \times 10^{-23} \times 9.1 \times 10^{-31}}{(6.626 \times 10^{-34})^2}\right)^{\frac{3}{2}} \times [300]^{\frac{3}{2}} \exp^{-\left(\frac{0.8 \times 1.602 \times 10^{19}}{2 \times 1.38 \times 10^{-23} \times 300}\right)}$$

$$N_i = 2.4 \times 10^{21} \times 5196.15 \times 1.22 \times 10^{-7} = 3.04 \times 10^{18} / m$$

5. The effective mass of the electron in silicon is 0.31 m_e, where m_e is the free electron mass. Find the electron concentration for silicon at 300 K assuming that the Fermi level lies exactly in the middle of the energy gap, given that the energy gap for silicon is 1.1 eV.

we know that,

Solutions,
$m_e^* = 0.31 m_e$
T=300K
if $E_v=0$, $E_c=E_g$ & also
$E_g = 1.1\text{eV} = 1.1\text{x}1.602\text{x}10^{-19}$J
here $E_F = E_g/2$ & $m_e^* = 0.31m_e$
$E_F = E_g/2$, N_e=?

$$N_e = 2\left(\frac{2\pi m_e^* kT}{h^2}\right)^{\frac{3}{2}} \times e^{-\left(\frac{E_c - E_F}{kT}\right)}$$

$$\therefore N_e = 2\left(\frac{2\pi m_e^* kT}{h^2}\right)^{\frac{3}{2}} \times e^{\frac{\left(-E_g + \frac{E_g}{2}\right)}{kT}} = 2\left(\frac{2\pi m_e^* kT}{h^2}\right)^{\frac{3}{2}} \times e^{\frac{(-E_g)}{2kT}}$$

$$N_e = 2\left(\frac{2\pi \times 9.1\times10^{-31}\times1.38\times10^{-23}\times300}{\left(6.626\times10^{-34}\right)^2}\right)^{\frac{3}{2}} \times e^{-\left(\frac{1.1\times1.602\times10^{-19}}{2\times1.38\times10^{-23}\times300}\right)}$$

$$N_e = 4.3255\times10^{24}\times4.784\times10^{-10} = 2.07\times10^{15}\,m^3$$

6. Calculate the position of the Fermi level for pure silicon at 300 K, if the electron concentration is $2\text{x}10^{15}\text{m}^{-3}$. Given that for silicon, the energy gap is 1.1eV, and the effective mass of electron is 0.31m_e, where m_e is the free electron mass.

we know that,

Solutions,
$N_e = 2\times10^{15}\,m^{-3}$
$E_g = 1.1\times1.602\times10^{-19}\,J$
$m_e^* = 0.31 m_e$
T=300 K
E_F=?

$$N_e = 2\left(\frac{2\pi m_e^* kT}{h^2}\right)^{\frac{3}{2}} \times e^{\left(\frac{E_F - E_g}{kT}\right)}$$

$$2\times10^{15} = 2\left(\frac{2\pi\times0.31\times9.1\times10^{-31}\times1.38\times10^{-23}\times300}{\left(6.626\times10^{-34}\right)^2}\right)^{\frac{3}{2}} \times e^{\left(\frac{E_F - E_g}{kT}\right)}$$

$$2\times10^{15} = 4.3255\times10^{24}\times e^{\left(\frac{E_F - E_g}{kT}\right)}$$

$$\text{or}\quad e^{\left(\frac{E_F - E_g}{kT}\right)} = \frac{2\times10^{15}}{4.3255\times10^{24}} = 4.624\times10^{-10}$$

$$\text{or}\quad \left(\frac{E_F - E_g}{kT}\right) = \ln\left(4.624\times10^{-10}\right) = -21.5$$

$$\therefore E_F - E_g = kT\times(-21.5) \qquad \text{or} \qquad E_F = E_g + kT\times(-21.5)$$

$$E_F = 1.1\times1.602\times10^{-19} - 1.38\times10^{-23}\times300\times(21.5)$$

$$E_F = 1.7622\times10^{-19} - 0.89\times10^{-19} \quad \text{or} \quad E_F = 0.872\times10^{-19}\,J$$

$$\text{or}\quad E_F = \frac{0.872\times10^{-19}}{1.602\times10^{-19}} = 0.544\,eV$$

The position of Fermi level is 0.544 eV.

7. The hall coefficient of a specimen is 1.25x10-4 m3/C and its resistivity is 9x10-4 Ωm. Find the mobility and the density of the charge carriers.

Solutions,

$$R_H = 1.25\times10^{-4} m^3/C$$
$$\rho = 9x10^{-4}\Omega m$$
$$n_h = ?$$
$$\mu_h = ?$$

$$W.K.T$$
$$R_H = \frac{1}{n_h e}$$
$$or \quad n_h = \frac{1}{R_H e}$$
$$n_h = \frac{1}{1.25x10^{-4}x1.602x10^{-19}}$$
$$n_h = 4.993x10^{22}/m^3$$

$$W.K.T$$
$$\rho_h = \frac{1}{n_h e \mu_h}$$
$$or \quad \mu_h = \frac{1}{n_e e \rho_h}$$
$$\mu_h = \frac{1}{4.993x10^{-4}x1.602x10^{-19}x9x10^{-4}}$$
$$n_h = 0.139\, m^3 V^{-1} s^{-1}$$

8. The hall co-efficient of a specimen of a doped silicon is found to be 3.66×10^{-4}m^3/C. The resistivity of the specimen is 9.93×10^{-3}ohm-m. Find the mobility and density of charge carrier, assuming single carrier concentration.

9. The conductivity and Hall co-efficient of an n type silicon specimen are 112 ohm^{-1}m^{-1} and 1.25×10^{-3} m^3/C, respectively. Calculate the charge carrier concentration and electron mobility.

10. Calculate the concentrations at which the acceptor atoms must be added to a germanium sample to get a p-type semiconductor with conductivity 0.15 ohm^{-1}m^{-1}. Given the mobility of holes 0.17 m^{2}V^{-1}s^{-1}.

Some important questions:

1. Explain the significance of Fermi level in an intrinsic semiconductor.
2. Explain the significance of Fermi levels in extrinsic semiconductor.
3. Explain the significance of Fermi level in n-type extrinsic semiconductor.
4. Explain the significance of Fermi level in p-type extrinsic semiconductor.
5. Derive the expression for electron concentration in an intrinsic semiconductor.

6. Derive the expression for hole concentration in an intrinsic semiconductor.
7. Derive the expression for electrical conductivity of an intrinsic semiconductor.
8. Derive an expression for resistivity of an intrinsic semiconductor.
9. Show that the Fermi level lies at the middle of the forbidden gap.
10. Show that $E_F = \frac{E_g}{2}$ or $E_F = \frac{E_C + E_V}{2}$
11. Derive an expression for intrinsic carrier concentration in an intrinsic semiconductor.
12. Explain the variation of resistance or resistivity with temperature in a semiconductor.
13. Define Hall effect. Derive an expression for Hall coefficient.
14. Define Hall effect. Derive an expression for Hall voltage.
15. Explain the construction and working of Laser diode.
16. Explain the construction and working of photo diode.
17. Describe how do you determine the resistivity of a semiconductor by Four probe method.
18. Write a note on phototransistor.

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